Application of EL methods on bivariate MRL and quantile correlation estimators

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1. Abstract

Based on the bivariate mean residual life (MRL) function proposed by Kulkarni and Rattihalli (2002), we apply the empirical likelihood (EL) and adjusted empirical likelihood (AEL) methods to the MRL function. The Wilk's theorem is established under general conditions. We profile the nuisance parameter in the EL and develop EL for the univariate MRL function. Extensive simulation studies show EL methods for both bivariate and one-dimensional MRL functions perform better than the normal approximation (NA) in terms of coverage probabilities. AEL methods results in noticeable better coverage probability. AEL method based on F-distribution calibration results in better coverage probability for small sample sizes. Two real datasets are used to illustrate the proposed procedure. We extend our study of EL methods by applying jackknife empirical likelihood (JEL) method to a quantile correlation and a quantile partial correlation function.

References

1. Kulkarni H.V., Rattihalli R. N. (2002) *Nonparametric estimation of a Bivariate Mean Residual Life Function.* Journal of the American Statistical Association
2. Chen, J., Variyath, M., Abraham, B. (2008) *Adjusted empirical likelihood and its properties.* Journal of Computational and Graphical Statistics, 17:426-443.
3. Li, G., Li, Y., Tsai, C. (2015) *Quantile Correlations and Quantile Autoregressive Modeling.* Journal of American Statistical Association

2. Introduction

- Kulkarni and Rattihalli (2002) proposed a bivariate MRL function.
- We apply an EL method to the proposed bivariate MRL estimator, profile the nuisance parameter and the EL method on one-dimensional, apply adjustment to the EL method, perform simulation study to compare coverage probability and confidence interval produced by the AEL, EL and NA methods for both the bivariate and one-dimensional after profiling.
- The proposed bivariate MRL function by Kulkarni and Rattihalli (2002) based on the empirical survival function for j=1 and j=2

$$\hat{m}_j(x,y) = \begin{cases} \frac{\sum (X_i - x) I[X_i > x, Y_i > y]}{\sum I[X_i > x, Y_i > y]} \\ \frac{\sum (Y_i - y) I[X_i > x, Y_i > y]}{\sum I[X_i > x, Y_i > y]} \end{cases}$$

3. Methods – Apply EL method

- We apply EL method to the bivariate MRL estimator as follows

$$L(\theta) = \sup\left\{ \prod_{i=1}^{n} (np_i) : p_1 \geq 0, \ldots, p_n \geq 0, \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i G(X_i;\theta) = 0 \right\}.$$

where $\theta = (m_1, m_2)$

Let

$$W_{ni} = (X_i - x) I(X_i > x, Y_i > y) - m_1(x,y) I(X_i > x, Y_i > y)$$

$$V_{ni} = (Y_i - y) I(X_i > x, Y_i > y) - m_2(x,y) I(X_i > x, Y_i > y)$$

Then

$$L(\beta_0) = sup\{\Pi p_i : \Sigma p_i = 1, \Sigma p_i W_{n,i} = 0, p_i V_{n,i} = 0, p_i \geq 0\}$$

$$R(\beta_0) = sup\{\Pi n p_i : \Sigma p_i = 1, \Sigma p_i W_{n,i} = 0, p_i V_{n,i} = 0, p_i \geq 0\}$$

3. Methods (Contd.)

Using Lagrange multipliers $R(\beta_0)$ is maximized when

$$p_i = \frac{1}{n}\{1 + \lambda_1 W_{n,i} + \lambda_2 V_{n,i}\}^{-1}$$

where λ_1 and λ_2 are solutions to the following equations

$$\frac{1}{n}\Sigma \frac{W_{n,i}}{1 + \lambda_1 W_{n,i} + \lambda_2 V_{n,i}} = 0$$

and

$$\frac{1}{n}\Sigma \frac{V_{n,i}}{1 + \lambda_1 W_{n,i} + \lambda_2 V_{n,i}} = 0$$

$$-2logR(\beta_0) \xrightarrow{D} \chi^2_2$$

4. Methods – Profile Nuisance parameter

- We profile the nuisance parameter using an algorithm proposed by in Zhao(2016).

$$\theta = (\alpha^{\tau}, \beta^{\tau})^{\tau} \qquad q = q_1 + q_2$$

where α and β are q_1 and q_2 dimensional parameters respectively, by showing that

$$-2logL(\alpha_0, \hat{\beta}(\alpha_0)) + 2logL(\hat{\theta}) \xrightarrow{D} \chi^2_{q_1}$$

where $L(\alpha, \beta) = L(\theta)$

$$\hat{\beta}(\alpha) = argmax_{\beta} L(\alpha, \beta)$$

a confidence region for α with level γ is constructed by

$$I^p_{\gamma} = \{\alpha : -2logL(\alpha, \hat{\beta}(\alpha)) \leq \chi^2_{q_1}(\gamma)\}.$$

5. Methods – Apply AEL method

We apply the AEL method proposed by Chen, Variyath and Abraham (2008) by setting

$$W_{n,n+1} = -(ln(n)/2) * \bar{W}_n$$

and $V_{n,n+1} = -(ln(n)/2) * \bar{V}_n$

6. Methods – F-distribution calibration

To improve performance for small sample sizes, we apply F-distribution calibration based EL ratio tests proposed by Owen (2001). For 2-dimension,

$$-2logR(\beta_0) \xrightarrow{D} 2(n-1)/(n-2)F_{2,n-2}(1-\alpha)$$

Similarly we also apply F-distribution after profiling the nuisance parameter. We apply to both EL and as well as AEL method.

7. Methods – Apply JEL method

Continuing our study of EL based methods, we apply JEL method to quantile correlation (qcor) and quantile partial correlation (qpcor) functions proposed by Li, Li and Tsai (2015). The pseudo-values for qcor is calculated as

$$W_n[j] = \frac{n * q\hat{c}or - ((n-1) * 1/n \sum_{i \neq j} \psi(Y_i - \hat{Q}_{\tau,Y})(X_i - \bar{X}))}{\sqrt{(\tau - \tau^2)\hat{\sigma}^2_X}}$$

8. Simulation Results- MRL

Using same setting as in Kulkarni and Rattihalli (2002), survival function based on Pareto distribution

$$S(x,y) = (x + y - 1)^{-a}, \quad x, y \geq 1$$

Table 1: 95% coverage probability $a = 6$

n	x	y	NA	EL	AEL	NA-1	EL-1	AEL-1
50	1.0	1.0	0.873	0.896	0.908	0.909	0.919	0.928
	1.0	1.09	0.862	0.867	0.892	0.905	0.910	0.923
	1.09	1.09	0.810	0.810	0.835	0.869	0.860	0.877
100	1.0	1.0	0.915	0.929	0.940	0.932	0.947	0.952
	1.00	1.09	0.896	0.912	0.929	0.914	0.922	0.924
	1.09	1.09	0.854	0.876	0.885	0.899	0.910	0.919

8. Simulation Results (Contd.)

Table 2: 95% coverage probability F-distribution calibration. $a = 6$

n	x	y	EL	EL-F	AEL	AEL-F	EL-1	EL-1-F
30	1.0	1.0	0.869	0.903	0.903	0.925	0.903	0.908
	1.0	1.09	0.799	0.835	0.839	0.869	0.851	0.863
	1.09	1.09	0.723	0.755	0.761	0.798	0.794	0.813
50	1.0	1.0	0.896	0.910	0.908	0.926	0.919	0.923
	1.00	1.09	0.867	0.890	0.892	0.901	0.910	0.917
	1.09	1.09	0.810	0.831	0.835	0.855	0.860	0.866

9. Application - Real Data

Table 3: Read dataset – Diabetic Retinopathy
Time to Blindness X= Treated Eye, Y =Control Eye

i	X_i	Y_i	i	X_i	Y_i
1	30.83	38.57	6	5.90	35.53
2	20.17	6.90	7	25.63	21.9
3	10.27	1.63	8	33.90	14.8
4	5.67	13.83	9	1.73	6.20
5	5.77	1.33	10	30.20	22.00

Data Source: R "SurvCorr" package. Original $n = 197$
Considered for study $n = 38$ uncensored observations

Table 4: Months of survival (Related to Table 3) – 95% CI length at observed pairs (X_i , Y_i)

		X				
Y		0	6	12	18	24
	NA	8.78	9.32	10.03	11.13	11.53
	EL	8.81	9.52	9.91	11.51	11.22
0	EL-F	9.19	9.87	10.29	11.97	11.64
	AEL	9.36	10.10	10.52	12.26	11.88
	AEL-F	9.69	10.47	10.98	12.67	12.45
	NA	15.02	15.09	15.09	14.49	14.49
	EL	15.14	15.42	15.42	13.52	13.51
12	EL-F	15.68	15.97	15.99	14.05	14.05
	AEL	16.06	16.32	16.36	14.44	14.43
	AEL-F	16.56	16.95	16.96	14.96	14.99

10. Simulation Results- qcor

Table 5: 95% coverage probability and CI of qcor estimate based on NA and JEL based methods

n	τ	NA	JEL	NA CI	JEL CI
50	0.25	0.946	0.962	0.469	0.521
	0.50	0.923	0.934	0.454	0.446
	0.75	0.926	0.957	0.477	0.531
100	0.25	0.941	0.963	0.328	0.365
	0.50	0.941	0.952	0.319	0.315
	0.75	0.94	0.969	0.334	0.369
200	0.25	0.958	0.975	0.230	0.254
	0.50	0.957	0.955	0.226	0.224
	0.75	0.948	0.972	0.232	0.257