Technical Complements to the paper

Shared Uncertainty in Measurement Error Problems, With Application to Nevada Test Site Fallout Data

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Web Appendix A: Sketch of Technical Arguments

A.1 Computation of $\Sigma_s^{-1}(\rho_s, \gamma_{\rm C})$

As is easily seen,

$$\begin{aligned}
\Sigma_s(\rho_s, \gamma_{\rm C}) &= (1-\gamma_{\rm C})\{(1-\rho_s)D_s + \rho_s v_s v_s^{\rm T}\};\\
D_s &= \mathrm{diag}(\sigma^2_{1s,\rm total}, ..., \sigma^2_{n_s s,\rm total});\\
v_s &= (\sigma_{1s,\rm total}, ..., \sigma_{n_s s,\rm total})^{\rm T};\\
\Sigma_s^{-1}(\rho_s, \gamma_{\rm C}) &= (1-\gamma_{\rm C})^{-1}\left\{(1-\rho_s)^{-1}D_s^{-1} - \frac{\rho_s}{(1-\rho_s)^2}\frac{D_s^{-1}v_s v_s^{\rm T} D_s^{-1}}{1+\rho_s(1-\rho_s)^{-1}v_s^{\rm T} D_s^{-1} v_s}\right\}\\
&= (1-\gamma_{\rm C})^{-1}(1-\rho_s)^{-1}\left\{D_s^{-1} - \frac{\rho_s}{1+(n_s-1)\rho_s} D_s^{-1} v_s v_s^{\rm T} D_s^{-1}\right\};\\
|\Sigma_s(\rho_s, \gamma_{\rm C})| &= (1-\gamma_{\rm C})^{n_s}|(1-\rho_s)D_s| \times |1+\rho_s(1-\rho_s)^{-1}v_s^{\rm T} D_s^{-1} v_s|\\
&= (1-\gamma_{\rm C})^{n_s}(1-\rho_s)^{n_s-1}\{1+(n_s-1)\rho_s\}\prod_{i=1}^{n_s}\sigma^2_{is,\rm total}.
\end{aligned}$$

A.2 Complete Conditionals for Gibbs Sampler

We will use Gibbs sampler to draw samples from the posterior distributions of X_{is}, L_{is}, ρ_s, $\gamma_{\rm C}$, β, μ_s, σ_s^2, θ. First we need to derive the complete conditional distributions for these variables and decide the sampling schemes from these complete conditionals. For those variables whose complete conditional distribution can not be easily generated, we will use a Metropolis step to update them. This algorithm is also referred as Metropolis-within-Gibbs algorithm.

- If we impose prior distribution Normal$(0, \sigma_{\mu_s}^2)$ to μ_s, then

$$\pi(\mu_s|\text{rest}) = \text{Normal}\left(\frac{\sigma_{\mu_s}^2}{\sigma_s^2 + n_s\sigma_{\mu_s}^2}\sum_{i=1}^{n_s} L_{is}, \frac{\sigma_s^2\sigma_{\mu_s}^2}{\sigma_s^2 + n_s\sigma_{\mu_s}^2}\right).$$

- Let $\pi(\sigma_s^2)$ be an Inverse Gamma distribution, i.e., $\pi(\sigma_s^2) \propto (\sigma_s^2)^{-(a_s+1)}\exp\{-1/(b_s\sigma_s^2)\}$, then

$$\pi(\sigma_s^2|\text{rest}) = IG\left[a_s + n_s/2, \left\{(1/2)\sum_{i=1}^{n_s}(L_{is} - \mu_s)^2 + b_s^{-1}\right\}^{-1}\right].$$

- Let $\mathbf{1}_{n_s}$ be a vector of n_s 1's, then the complete conditional for $\mathcal{L}_s = (L_{1s}, \cdots, L_{n_s s})^{\mathrm{T}}$ is

$$\begin{aligned}
\pi(\mathcal{L}_s|\text{rest}) &\propto \exp\left\{-\frac{1}{2\sigma_s^2}(\mathcal{L}_s - \mu_s\mathbf{1}_{n_s})^{\mathrm{T}}(\mathcal{L}_s - \mu_s\mathbf{1}_{n_s}) - \frac{1}{2\gamma_{\mathrm{C}}}(\mathcal{L}_s - \mathcal{W}_s)^{\mathrm{T}}D_s^{-1}(\mathcal{L}_s - \mathcal{W}_s)\right. \\
&\qquad \left. -\frac{1}{2}(\mathcal{L}_s - \mathcal{X}_s)^{\mathrm{T}}\Sigma_s^{-1}(\rho_s, \gamma_{\mathrm{C}})(\mathcal{L}_s - \mathcal{X}_s)\right\} \\
&\propto \exp\left[-\frac{1}{2}\mathcal{L}_s^{\mathrm{T}}\{\sigma_s^{-2}I + \gamma_{\mathrm{C}}^{-1}D_s^{-1} + \Sigma_s^{-1}(\rho_s, \gamma_{\mathrm{C}})\}\mathcal{L}_s\right. \\
&\qquad \left. +\mathcal{L}_s^{\mathrm{T}}\{\mu_s\sigma_s^{-2}\mathbf{1}_{n_s} + \gamma_{\mathrm{C}}^{-1}D_s^{-1}\mathcal{W}_s + \Sigma_s^{-1}(\rho_s, \gamma_{\mathrm{C}})\mathcal{X}_s\}\right] \\
&= \text{Normal}(\Omega_s^{-1}h_s, \Omega_s^{-1}), \qquad \text{(A.1)}
\end{aligned}$$

where $\Omega_s = \sigma_s^{-2}I + \gamma_{\mathrm{C}}^{-1}D_s^{-1} + \Sigma_s^{-1}(\rho_s, \gamma_{\mathrm{C}})$, $h_s = \mu_s\sigma_s^{-2}\mathbf{1}_{n_s} + \gamma_{\mathrm{C}}^{-1}D_s^{-1}\mathcal{W}_s + \Sigma_s^{-1}(\rho_s, \gamma_{\mathrm{C}})\mathcal{X}_s$.

- We let the prior for β to be Normal$(0, \sigma_\beta^2 I)$, then the complete conditional for β is

$$\pi(\beta|\text{rest}) \propto \prod_{i,s}\frac{(\exp[Z_{is}^{\mathrm{T}}\beta + \log\{1 + \theta\exp(X_{is})\}])^{Y_{is}}}{1 + \exp[Z_{is}^{\mathrm{T}}\beta + \log\{1 + \theta\exp(X_{is})\}]}\exp\left\{-\frac{1}{2\sigma_\beta^2}\beta^T\beta\right\}.$$

We use a Metropolis step to update β. Denote the current value of β as β_{curr}, then generate a candidate value of β by kernel $q(\beta_{curr}, \beta_{new}) = \text{Normal}(\beta_{curr}, 0.01I)$. The new value is accepted with probability

$$\alpha = \min\left\{1, \frac{\pi(\beta = \beta_{new}|\text{rest})}{\pi(\beta = \beta_{curr}|\text{rest})}\right\}.$$

- Complete conditional for γ_{C}:

$$\pi(\gamma_{\mathrm{C}}|\text{rest}) \propto I(a_c \le \gamma_{\mathrm{C}} \le b_c)\gamma_{\mathrm{C}}^{-N/2}\exp\left\{-\frac{1}{2\gamma_{\mathrm{C}}}\sum_{i,s}(W_{is} - L_{is})^2/\sigma_{is,\text{total}}^2\right\}$$

$$\begin{aligned}
&\times \prod_s |\Sigma_s(\rho_s,\gamma_{\mathrm{C}})|^{-1/2} \times \exp\left\{-\frac{1}{2}\sum_s (\mathcal{X}_s - \mathcal{L}_s)^{\mathrm{T}} \Sigma_s^{-1}(\rho_s,\gamma_{\mathrm{C}})(\mathcal{X}_s-\mathcal{L}_s)\right\} \\
\propto\ & I(a_c \le \gamma_{\mathrm{C}} \le b_c)\gamma_{\mathrm{C}}^{-N/2}(1-\gamma_{\mathrm{C}})^{-N/2}\exp\left\{-\frac{1}{2\gamma_{\mathrm{C}}}\sum_{i,s}(W_{is}-L_{is})^2/\sigma^2_{is,\mathrm{total}}\right\} \\
&\times \exp\left[-\frac{1}{2(1-\gamma_{\mathrm{C}})}\sum_s(\mathcal{X}_s-\mathcal{L}_s)^{\mathrm{T}}\{(1-\rho_s)D_s+\rho_s v_s v_s^{\mathrm{T}}\}^{-1}(\mathcal{X}_s - \mathcal{L}_s)\right].
\end{aligned}$$

Again, we use a Metropolis step to update γ_{C}. A new value $\gamma_{\mathrm{C,new}}$ is generated from Uniform$[a_c, b_c]$. We accept the new value with probability

$$\alpha = \min\left\{1, \frac{\pi(\gamma_{\mathrm{C}} = \gamma_{\mathrm{C,new}}|\mathrm{rest})}{\pi(\gamma_{\mathrm{C}} = \gamma_{\mathrm{C,curr}}|\mathrm{rest})}\right\}.$$

- Complete conditional for ρ_s:

$$\begin{aligned}
\pi(\rho_s|\mathrm{rest}) \ \propto\ & I(c_s \le \rho_s \le d_s)|\Sigma_s(\rho_s,\gamma_{\mathrm{C}})|^{-1/2}\exp\left\{-\frac{1}{2}(\mathcal{X}_s-\mathcal{L}_s)^{\mathrm{T}}\Sigma_s^{-1}(\rho_s,\gamma_{\mathrm{C}})(\mathcal{X}_s-\mathcal{L}_s)\right\} \\
\propto\ & I(c_s \le \rho_s \le d_s)(1-\rho_s)^{-(n_s-1)/2}\{1+(n_s-1)\rho_s\}^{-1/2}\exp\Bigg[-\frac{1}{2}(1-\gamma_{\mathrm{C}})^{-1} \\
&\times(1-\rho_s)^{-1}(\mathcal{X}_s-\mathcal{L}_s)^{\mathrm{T}}\left\{D_s^{-1} - \frac{\rho_s}{1+(n_s-1)\rho_s}D_s^{-1}v_s v_s^{\mathrm{T}} D_s^{-1}\right\}(\mathcal{X}_s-\mathcal{L}_s)\Bigg].
\end{aligned}$$

We use a Metropolis step similar to that of γ_{C} to update ρ_s. The candidate value is generated from Uniform$[c_s, d_s]$.

- Complete conditional for θ:

$$\begin{aligned}
\pi(\theta|\mathrm{rest}) \ \propto\ & \pi(\theta)\prod_{i,s}(H[Z_{is}^{\mathrm{T}}\beta + \log\{1+\theta\exp(X_{is})\}])^{Y_{is}} \\
&\times(1-H[Z_{is}^{\mathrm{T}}\beta+\log\{1+\theta\exp(X_{is})\}])^{1-Y_{is}}.
\end{aligned}$$

Again, θ is updated by a Metropolis step similar to above. We consider the following two schemes.

(θ.1) For the truncated normal prior, we use the prior $\pi(\theta)$ as the proposal density. A new value θ_{new} is generated from $\pi(\cdot)$, and then it is determine to be accepted with probability

$$\alpha = \min\left\{1, \frac{\mathcal{P}_\theta(\theta_{\mathrm{new}})}{\mathcal{P}_\theta(\theta_{\mathrm{curr}})}\right\},$$

where $\mathcal{P}_\theta(t) = \prod_{i,s}(H[Z_{is}^{\mathrm{T}}\beta + \log\{1+t\exp(X_{is})\}])^{Y_{is}}(1-H[Z_{is}^{\mathrm{T}}\beta+\log\{1+t\exp(X_{is})\}])^{1-Y_{is}}$. In the analysis, we used proper priors for θ, such as the Truncated Normal priors.

(θ.2) For the approximate Jeffreys prior, we used a transition kernel $q(\theta_{\text{curr}}, \theta_{\text{new}}) = TN(\theta_{\text{curr}}, \sigma_q^2)$, where $TN(\mu, \sigma^2)$ is the truncated normal distribution with density function $I(x \geq 0)\phi\{(x-\mu)/\sigma\}/\Phi(\mu/\sigma)$, ϕ and Φ are the density and distribution function of a standard normal random variable. Then the new value θ_{new} is accepted with probability

$$\alpha = \min\left\{1, \frac{\pi(\theta = \theta_{\text{new}}|\text{rest})/\Phi(\theta_{\text{new}}/\sigma_q)}{\pi(\theta = \theta_{\text{curr}}|\text{rest})/\Phi(\theta_{\text{curr}}/\sigma_q)}\right\}.$$

A.3 Method of Scoring for EM Algorithm

The method of scoring in Section 4.2 is

$$\widehat{\Theta}_1^{(k+1)} = \widehat{\Theta}_1^{(k)} - A^{-1}(\Theta_1^{(k)})B(\Theta_1^{(k)}),$$

where $A(\Theta_1) = \partial^2\widehat{Q}/\partial\Theta_1\partial\Theta_1^{\mathrm{T}}$ and $B(\Theta_1) = \partial\widehat{Q}/\partial\Theta_1$, these terms given as

$$\begin{aligned}
P_{is}(\beta, \delta, X_{is}^{(b)}) &= H[Z_{is}^{\mathrm{T}}\beta + \log\{1+\exp(\delta + X_{is}^{(b)})\}], \\
\frac{\partial\widehat{Q}}{\partial\beta} &= B^{-1}\sum_b\sum_{s,i}\{Y_{is} - P_{is}(\beta,\delta,X_{is}^{(b)})\}Z_{is}, \\
\frac{\partial\widehat{Q}}{\partial\delta} &= B^{-1}\sum_b\sum_{s,i}\{Y_{is} - P_{is}(\beta,\delta,X_{is}^{(b)})\}\frac{\exp(\delta + X_{is}^{(b)})}{1+\exp(\delta + X_{is}^{(b)})}, \\
\frac{\partial^2\widehat{Q}}{\partial\beta\partial\beta^{\mathrm{T}}} &= -B^{-1}\sum_b\sum_{s,i}P_{is}(\beta,\delta,X_{is}^{(b)})\{1-P_{is}(\beta,\delta,X_{is}^{(b)})\}Z_{is}Z_{is}^{\mathrm{T}}, \\
\frac{\partial^2\widehat{Q}}{\partial\beta\partial\delta} &= -B^{-1}\sum_b\sum_{s,i}P_{is}(\beta,\delta,X_{is}^{(b)})\{1-P_{is}(\beta,\delta,X_{is}^{(b)})\}\frac{\exp(\delta + X_{is}^{(b)})}{1+\exp(\delta + X_{is}^{(b)})}Z_{is}, \\
\frac{\partial^2\widehat{Q}}{\partial\delta^2} &= -B^{-1}\sum_b\sum_{s,i}P_{is}(\beta,\delta,X_{is}^{(b)})\{1-P_{is}(\beta,\delta,X_{is}^{(b)})\}\left\{\frac{\exp(\delta + X_{is}^{(b)})}{1+\exp(\delta + X_{is}^{(b)})}\right\}^2.
\end{aligned}$$

A.4 Formulae For Computing (8)

Let $P_{is}(\beta, \delta, x) = H[Z_{is}^{\mathrm{T}}\beta + \log\{1+\exp(\delta + x)\}]$. Then

$$\frac{\partial \log f_1}{\partial\beta} = \sum_{s,i}\{Y_{is} - P_{is}(\beta,\delta,X_{is})\}Z_{is},$$

$$
\begin{aligned}
\frac{\partial \log f_1}{\partial \delta} &= \sum_{s,i}\{Y_{is} - P_{is}(\beta,\delta,X_{is})\}\frac{\exp(\delta + X_{is})}{1+\exp(\delta+X_{is})},\\
\frac{\partial \log f_1}{\partial \mu_s} &= (\sigma_s^2)^{-1}\sum_i (L_{is}-\mu_s),\\
\frac{\partial \log f_1}{\partial \sigma_s^2} &= (2\sigma_s^4)^{-1}\sum_i (L_{is}-\mu_s)^2 - n_s/(2\sigma_s^2),\\
-\frac{\partial^2 \log f_1}{\partial \beta \partial \beta^{\mathrm{T}}} &= \sum_{s,i} P(\beta,\delta,X_{is})\{1-P(\beta,\delta,X_{is})\} Z_{is} Z_{is}^{\mathrm{T}},\\
-\frac{\partial^2 \log f_1}{\partial \beta \partial \delta} &= \sum_{s,i} P(\beta,\delta,X_{is})\{1-P(\beta,\delta,X_{is})\}\frac{\exp(\delta+X_{is})}{1+\exp(\delta+X_{is})} Z_{is},\\
-\frac{\partial^2 \log f_1}{\partial \delta^2} &= -\sum_{s,i}\{Y_{is}-P_{is}(\beta,\delta,X_{is})\}\frac{\exp(\delta+X_{is})}{\{1+\exp(\delta+X_{is})\}^2}\\
&\qquad + \sum_{s,i} P(\beta,\delta,X_{is})\{1-P(\beta,\delta,X_{is})\}\left\{\frac{\exp(\delta+X_{is})}{1+\exp(\delta+X_{is})}\right\}^2,\\
-\frac{\partial^2 \log f_1}{\partial \mu_s \partial \mu_{s'}} &= I(s=s') n_s/\sigma_s^2,\\
-\frac{\partial^2 \log f_1}{\partial \sigma_s^2 \partial \sigma_{s'}^2} &= I(s=s')\left\{\sigma_s^{-6}\sum_i (L_{is}-\mu_s)^2 - n_s/(2\sigma_s^4)\right\},\\
-\frac{\partial^2 \log f_1}{\partial \mu_s \partial \sigma_{s'}^2} &= I(s=s')\left\{\sigma_s^{-4}\sum_i (L_{is}-\mu_s)\right\}.
\end{aligned}
$$

Web Figures W.1 and W.2

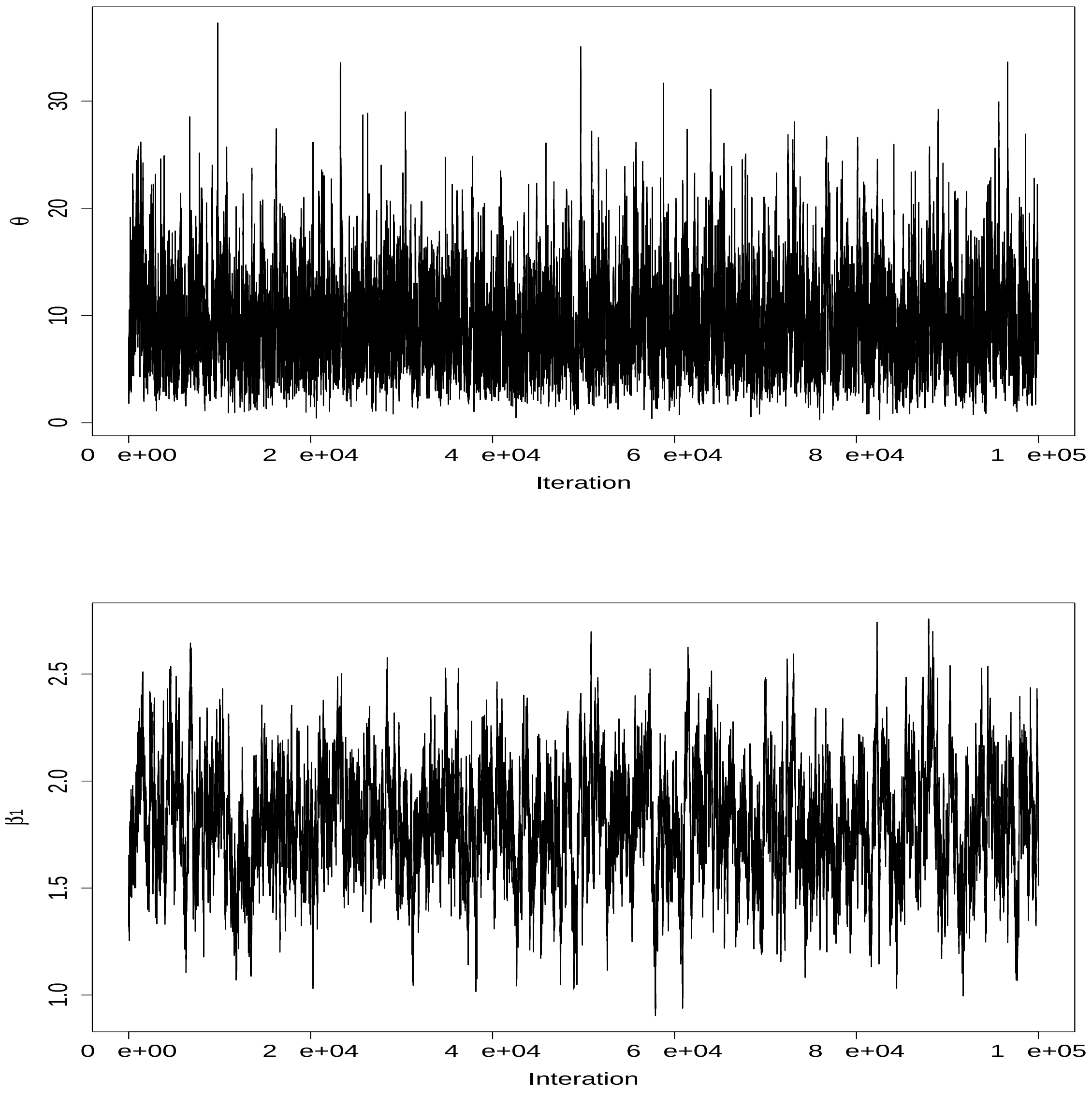

Figure W.1: History for the excess relative risk parameter θ per gray and the gender effect β_1 for thyroiditis, in the case of shared Berkson uncertainties and using the approximate Jeffreys prior.

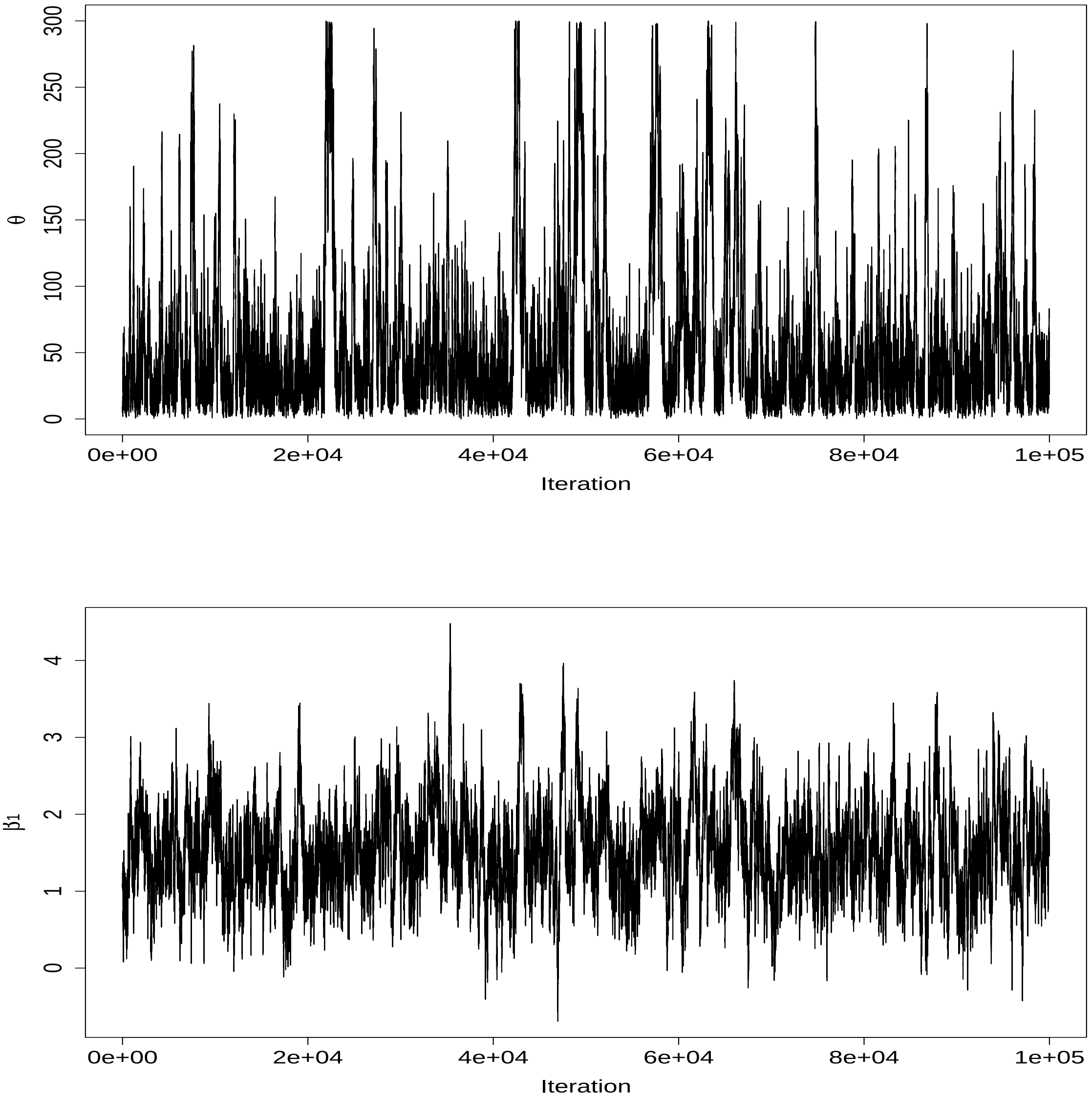

Figure W.2: History for the excess relative risk parameter θ per gray for neoplasm, in the case of shared Berkson uncertainties and using the approximate Jeffreys prior.