Optimal Designs for Accelerated Life Testing Experiments with Step-stress Plans

Oct. 19, 2012

DAE 2012, Athens

Xiaojian Xu

(Joint work with Scott Hunt, University of Guelph)

Brock University

Outline

- Introduction
- Problems of Interest
- Optimal designs for a Weibull life distribution with a non-constant shape parameter
- Robust designs against imprecision in a stress-life relationship
- Future study

Introduction

Accelerated Life Testing (ALT)

- For highly reliable products or components, a life testing experiment takes too long to observe any failures under normal operating conditions.

- ALT is often used to shorten the life so that the failures can be quickly obtained in a reasonable time period.

- ALT experiments are normally conducted at stress levels which are higher than normal use stress level, S_0.

Accelerated Life Testing (ALT)

The commonly used methods of stress loading (acceleration) include:

- Constant stress ALT: each unit is subjected to an accelerated stress level and this level remains unchanged during the testing period although different units may be under different stress levels.

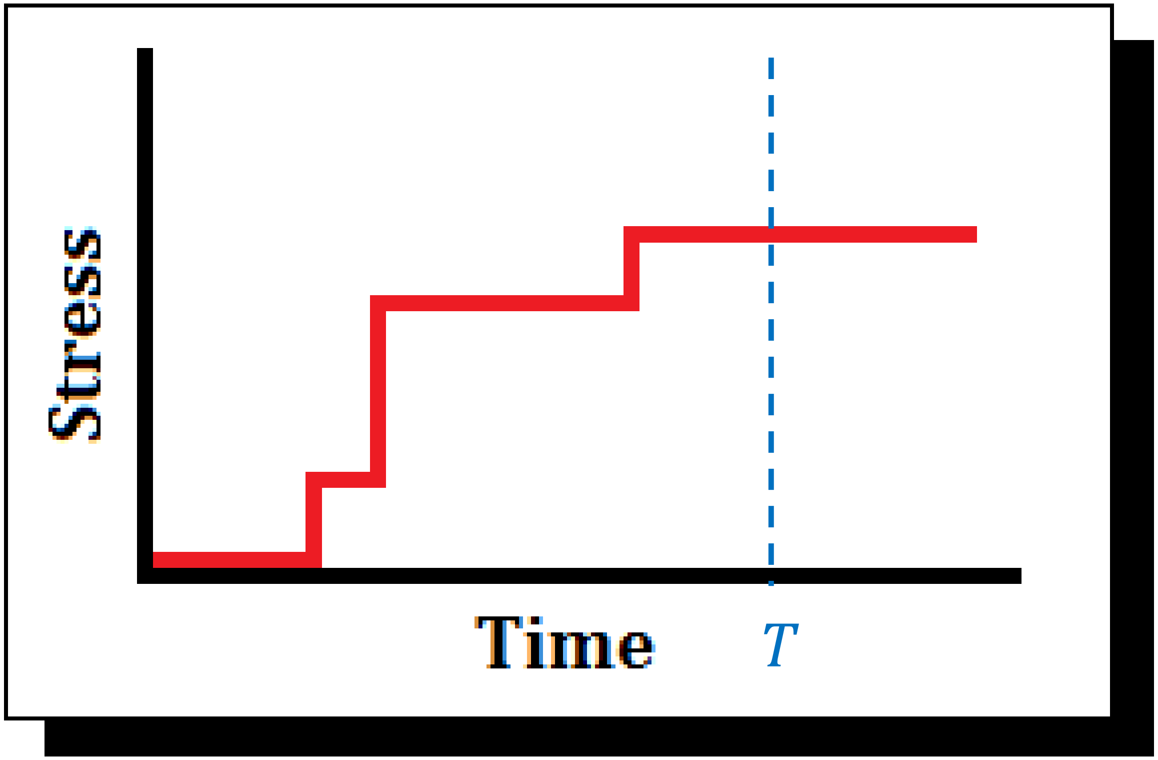
- Step-stress ALT: the stress subjected to each test unit is not constant but is changing in a stepwise manner.

- Compare to constant stress ALTs, step-stress ALTs often obtain information much more quickly.

Step-Stress ALT

- Stress level varies with time.
- All test units are tested at an accelerated stress level (all at the same level); at the first stress changing time, all of the surviving units are moved to a higher stress level; at the second stress changing time, all of the surviving units are moved to an even higher stress level, and so on, until all units fail (complete data) or until a specified censoring time (censored data).
- A simple step-stress only use two accelerated stress levels with one stress-changing time.

Step-Stress ALT



Graphical representation of a 4-step-stress plan

Problems of Interest

Two Stress Change Effect Models

For a step-stress plan, the effect of the change in stress levels on the remaining life-time of a product needs to be explained.

With a k-step-stress model, two commonly used ones are:

1. Cumulative Exposure Model (CE model) (Nelson, 1980)

For a fixed stress level x_i, the cumulative distribution function $F_i(t)$.

This model ensures the cumulative function for $0 \le t \le T$ be continuous with

$$F(t) = \begin{cases} F_1(t), 0 \le t \le \tau_1; \\ F_2(r_1 + (t - \tau_1)),\ \tau_1 < t \le \tau_2; \\ \cdots \\ F_k(r_{k-1} + (t - \tau_{k-1})),\ \tau_{k-1} < t \le T. \end{cases}$$

where r_i satisfies

$$F_i(r_{i-1}) = F_{i-1}(r_{i-2} + (\tau_{i-1} - \tau_{i-2})).$$

A KH Model

2. Khamis-Higgins Model (KH model) (Khamis and Higgins, 1998) with a Weibull life distribution:

For a scale parameter at x_i being θ_i,the hazard funtion is $h(\theta_i)$, when a shape parameter stays constant:

$$h(t) = \begin{cases} h(t|\theta_1), 0 \leq t \leq \tau_1; \\ \alpha_1 h(t|\theta_2), \ \tau_1 < t \leq \tau_2; \\ \dots \\ \alpha_{k-1} h(t|\theta_k), \ \tau_{k-1} < t \leq T. \end{cases}$$

Often, $\alpha_i = \frac{\theta_1}{\theta_i}$ is used.

Previous work on optimal designs for step-stress ALTs

Under a Cumulative Exposure Model (CE model):

- Miller and Nelson (1983) and Bai, Kim, and Kee (1989) discuss the optimal simple step-stress plans when the lifetimes have an exponential distribution.

- Bai and Kim (1993) also obtain an optimal simple step-stress plan for a Weibull lifetime distribution for censored data.

- Ma and Meeker (2008) extend the optimal step-stress ALT plans construction to the general log-location-scale distributions.

Previous work on optimal designs for step-stress ALTs

Under the KH model with a Weibull life distribution:

- Alhadeed and Yang (2002) derived the optimal simple step-stress .

- Most recently, Fard and Li (2009) investigated the optimal simple step-stress ALT design for reliability prediction.

- We will consider the Khamis-Higgins model too.

-- All of the above results are under the assumption that the Weibull shape parameter doesn’t change when a stress level changes.

Motivation 1

- However, as indicated in Nelson (2004), for many products, the spread of lifetime (or transformed lifetime) is also a function of stress.

- Cox and Oakes (2002) pointed out that the accelerated stress not only affects the scale parameter, but the shape parameter of the Weibull distribution as well.

- Such behaviour in lifetime should be taken into account in the design stage of the ALT experiments.

- We study the optimal designs for step-stress when both scale and shape parameters of a Weibull life distribution are function of the stress.

Previous work on robust designs for constant-stress ALTs

- Ginebra and Sen (1998) investigate the minimax ALT when the parameter values, which the design depends on, are possibly misspecified.

- Pascual and Montepiedra (2003) discuss the robust designs when there is uncertainty on lifetime distribution.

- Pascual (2006) develop the model-robust designs against imprecision in the assumed life-stress relationship.

- All these paper consider the robust designs for the constant-stress ALTs.

Motivation 2

- Minimal research has been conducted in robust designs against model departure for step-stress ALT .
- Pascual (2006) noted that estimates under the linear and quadratic models may appear similar within the stress range of the life tests, but tend to diverge beyond this range.
- This is certainly problematic for the practitioner because the results from the fitted regression model at accelerated levels are extrapolated to normal operating conditions.

Problems

Assuming a Weibull life distribution,

1. Optimal designs for step- stress ALT with a non-constant shape parameter, and

2. Robust designs for step-stress ALT when the scale-stress relastionship is possibly misspecified.

Notation

- n – number of initial test units
- S_0 – use stress level
- S_1 – low accelerated stress level
- S_2 – high accelerated stress level
- τ – stress changing time
- T – censoring time

Optimal Designs

Model Assumptions

Simple step-stress plan:

- $S_0<S_1<S_2$

- The failure time of a test unit:

$$t_{ij} \overset{\text{i.i.d.}}{\sim} Weibull(\theta_i,\delta_i),\ i=0,1,2$$

- The scale parameter θ_i, i=0,1,2 is a log-linear function of stress:

$$ln(\theta_i)=\beta_0+\beta_1 S_i\ ,$$

where β_0 and β_1 are unknown parameters.

For Complete Data

All test units are subjected to S_1 until τ, at which the stress is raised to $S_{2,}$ the test is continued until all the test units have failed.

$$y_{ij} = \ln(t_{ij}) \overset{\text{i.i.d.}}{\sim} S.E.V.(\mu_i, \sigma_i),\ i=0,1,2,$$

where

$$\mu_i = \beta_0 + \beta_1 S_i\,,$$

and

$$\sigma_i = 1/\delta_{i.}$$

For Censored Data

Let

$$y_{ij} = ln\left(\frac{t_{ij}}{T}\right)$$

The model can be simplified:

$$y_{ij} \overset{\text{i.i.d.}}{\sim} S.E.V.(\mu_i, \sigma_i), \ i=0,1,2,$$

where

$$\mu_i = \beta_0 + \beta_1 S_i - ln(T),$$

and

$$\sigma_i = 1/\delta_i.$$

Let

$$\tau_0 = \frac{\tau}{T}, \text{ and } C_i = \frac{\ln(\tau_0) - \mu_i}{\sigma_i}.$$

Heteroscedasticity

Let the transformed stress $x_i = \frac{S_i - S_0}{S_2 - S_0}$.

Assume the shape-stress relationship is:

$$\sigma_i^2 \;=\; \sigma_0^2 + \gamma x_i$$

where $\gamma \geq 0$, and $\sigma_0{}^2$ is the reciprocal of the squared Weibull shape parameter at use stress level.

Design Criterion

We choose the optimal stress-changing time so that nAVAR of the MLE of reliability, at the use stress level and at a given time, can be minimized.

Reliability Estimate

The parameter of interest, the reliability at time φ for censored data under the use stress level S_0 is

$$R_{S_0}(\varphi) = exp\left(-exp\left(\frac{ln\left(\frac{\varphi}{T}\right) - ln\left(\frac{\theta_0}{T}\right)}{\sigma_0}\right)\right).$$

By the invariance property of MLE, the MLE of reliability at time φ under the use stress level S_0 is

$$\widehat{R_{S_0}(\varphi)} = exp\left(-exp\left(\frac{ln\left(\frac{\varphi}{T}\right) - \frac{\widehat{\mu_1} - x_1\widehat{\mu_2}}{1 - x_1}}{\sigma_0}\right)\right),$$

where $\widehat{\mu_1}$ and $\widehat{\mu_2}$ are the MLEs of μ_1 and μ_2.

Note: For complete data, using T=1 for the derivation.

Optimization Criterion

The asymptotic variance of the MLE of reliability is

$$AVAR\left[\widehat{R_{S_0}(\varphi)}\right] = AVAR\left[exp\left(-exp\left(\frac{ln\left(\frac{\varphi}{T}\right)-\frac{\widehat{\mu_1}-x_1\widehat{\mu_2}}{1-x_1}}{\sigma_0}\right)\right)\right] = H^T \cdot F^{-1} \cdot H,$$

where F is the expected Fisher information matrix and H is defined as

$$H = \begin{bmatrix} \frac{\partial \widehat{R_{S_0}(\varphi)}}{\partial \widehat{\mu_1}} & \frac{\partial \widehat{R_{S_0}(\varphi)}}{\partial \widehat{\mu_2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sigma_0(1-x_1)} exp\left(-exp\left(\frac{ln\left(\frac{\varphi}{T}\right)-\frac{\widehat{\mu_1}-x_1\widehat{\mu_2}}{1-x_1}}{\sigma_0}\right)\right) exp\left(\frac{ln\left(\frac{\varphi}{T}\right)-\frac{\widehat{\mu_1}-x_1\widehat{\mu_2}}{1-x_1}}{\sigma_0}\right) \\ -\frac{x_1}{\sigma_0(1-x_1)} exp\left(-exp\left(\frac{ln\left(\frac{\varphi}{T}\right)-\frac{\widehat{\mu_1}-x_1\widehat{\mu_2}}{1-x_1}}{\sigma_0}\right)\right) exp\left(\frac{ln\left(\frac{\varphi}{T}\right)-\frac{\widehat{\mu_1}-x_1\widehat{\mu_2}}{1-x_1}}{\sigma_0}\right) \end{bmatrix}.$$

Design Criterion (Censored)

- The expected Fisher information matrix F at the use stress level S_0 can be derived as

$$F = \frac{n}{{\sigma_0}^2} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $$A_{11} = \frac{{\sigma_0}^2}{{\sigma_1}^2}\left[1 - \exp\left(-exp(\widehat{C_1})\right)\right],\ A_{12} = A_{21} = 0,$$

$$A_{22} = \frac{{\sigma_0}^2}{{\sigma_2}^2}\left[\exp\left(-exp(\widehat{C_1})\right) - exp\left(-exp(\widehat{C_T}) + exp(\widehat{C_2}) - exp(\widehat{C_1})\right)\right],$$

$$\text{with } \hat{C}_i = \frac{\ln(\tau_0) - \widehat{\mu}_i}{\sigma_i},\ i = 1, 2,\ \text{and } \widehat{C_T} = -\frac{\widehat{\mu_2}}{\sigma_2}.$$

Design Criterion (Complete)

- The elements of the Fisher information matrix for complete data are:

$$A_{11}=\frac{{\sigma_0}^2}{{\sigma_1}^2}\left[1-\exp\left(-exp(\widehat{C_1})\right)\right],\ A_{12}=A_{21}=0,\ A_{22}=\frac{{\sigma_0}^2}{{\sigma_2}^2}\left[\exp\left(-exp(\widehat{C_1})\right)\right],$$

Example

- A simple step-stress test for cable insulation is run to estimate the reliability at φ=2000 minutes, under a normal use voltage: $S_0 = 20kV$.

- Test stress levels are: $S_1 = 24kV$, $S_2 = 30kV$. *(so,* $x_1 = 0.4$).

- n=100 test units, T=1000 minutes

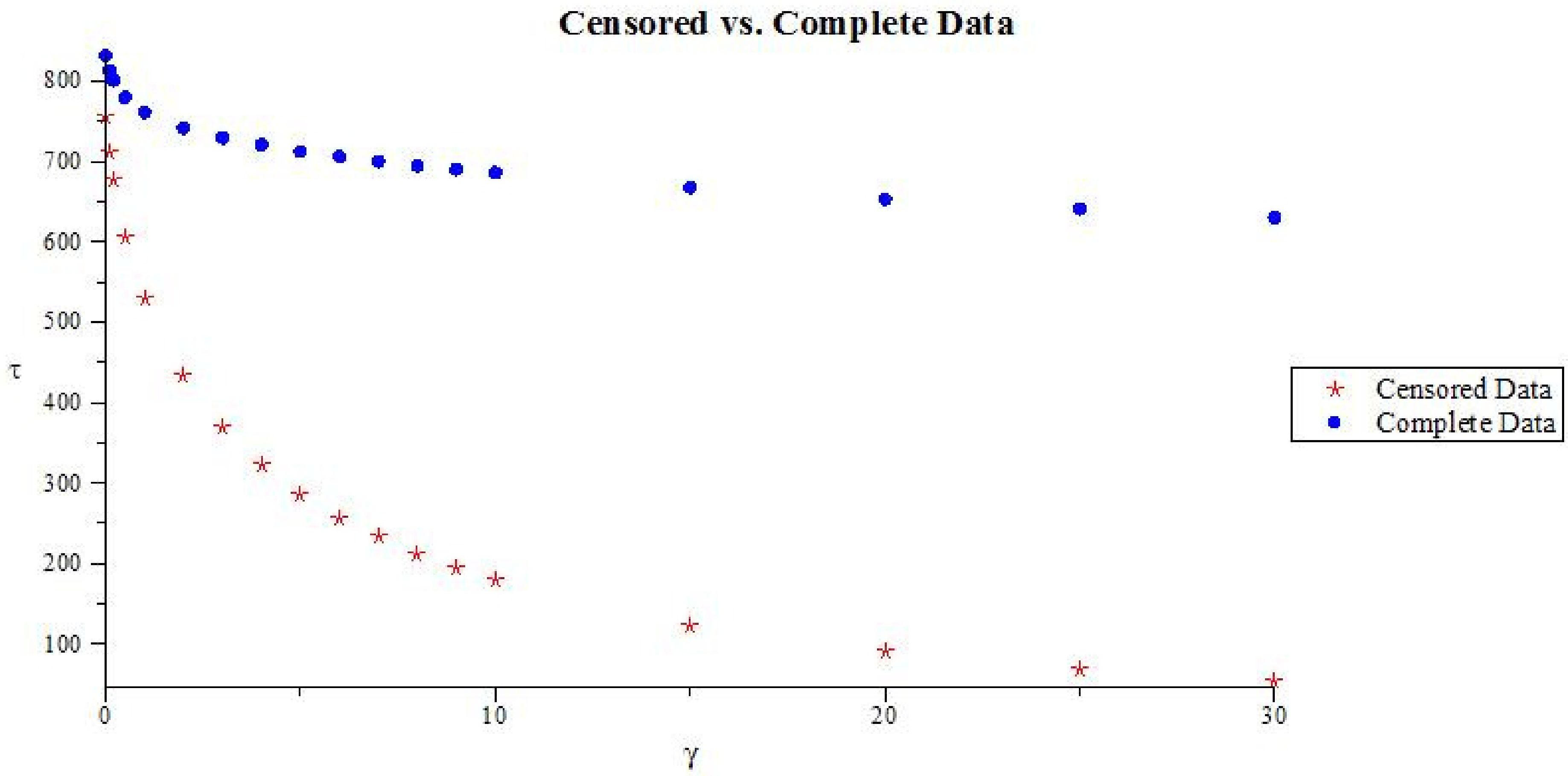
- *Assume* $\delta_0 = 2.2$, then $\sigma_i^2 = \frac{1}{4.84} + \gamma x_i$ *i=0,1,2*

- *With* $\widehat{\theta}_1 = 750$ and $\widehat{\theta}_2 = 600$, we have the optimal stress changing times for both cases of complete and censored data.

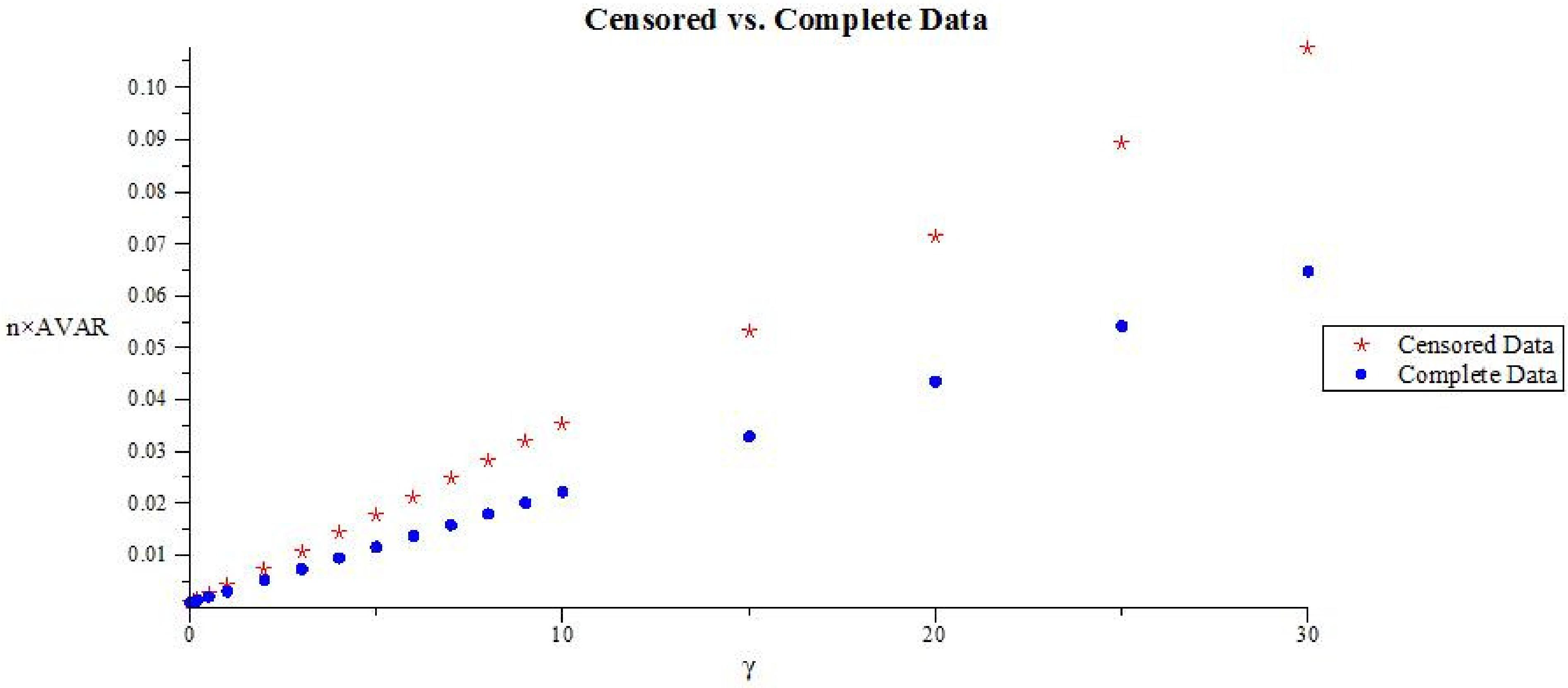
Optimal Stress-Changing Times

	Complete		Censored $(T = 1000)$	
γ	min. $n \times AVAR$	Optimal τ	min. $n \times AVAR$	Optimal τ
0	0.000810	831.9	0.000888	755.5
0.01	0.000832	828.7	0.000918	750.4
0.1	0.001031	813.0	0.001183	711.3
0.2	0.001249	801.1	0.001484	677.8
0.5	0.001896	780.0	0.002414	606.4
1	0.002967	761.5	0.004021	529.2
2	0.005162	741.7	0.007348	432.7
5	0.011499	712.5	0.017685	286.7
10	0.022156	686.0	0.035355	178.8
15	0.032813	667.7	0.053253	124.2
20	0.043469	653.2	0.071273	91.2
25	0.054125	640.9	0.089371	69.5
30	0.064781	630.1	0.107525	54.3

Stress-Changing Time Comparison



*n*AVAR Comparison



Discussion

- The model considered is more general, as a special case of $\gamma=0$ creates a constant shape parameter.

- Since the mean lifetime of the test units is decreasing as γ increases, there are more expected failures at the lower stress level.

- So the optimal changing stress time occurs at an earlier time. This reveals that

- As γ increases the optimal τ decreases.

Robust Design Against Possible Departure from the Stress-Life Relationship

Robust Planning

- Assumes a particular model to fit the data, but recognizes the possibility that the assumed model may not be precise.
- Find the optimal stress-changing time so that the Asymptotic MSE can be minimized.

Model Assumptions

Modify the step-stress ALT model used earlier.

$$t_{ij} \overset{iid}{\sim} Weibull(\theta_i, \delta),\ i=0,1,2$$

Two scale stress relationships are considered as possible candidates: linear and quadratic.

Suppose the linear model is fitted,

$$\ln(\theta_{li})=\beta_0+\beta_1 S_{i;}$$

The true model is actually quadratic,

$$\ln(\theta_{qi})=\alpha_0+\alpha_1 S_i+\alpha_2 S_i^2$$

Asymptotic Distribution of MLEs

- M_q -(quadratic model) true model
- M_l -(linear model) the fitted model

- $L(\beta,\xi)$ - the log-likelihood under M_l
- $L(\alpha,\xi)$ - the log-likelihood under M_q

- $I(\alpha: \beta) = E_{Mq}[L(\alpha,\xi)-L(\beta,\xi)]$,

  where $\alpha=[\alpha_0,\alpha_1,\alpha_2]^T$ and $\beta=[\beta_0,\beta_1]^T$

- Fix α and let β^* be the value of β that minimizes $I(\alpha: \beta)$.

Asymptotic Distribution of MLEs

Define the matrices

$$A(\beta) = \begin{bmatrix} E_{M_q}\left(\frac{\partial^2 \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_0^2}\right) & E_{M_q}\left(\frac{\partial^2 \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_0 \partial \beta_1}\right) \\ E_{M_q}\left(\frac{\partial^2 \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_1 \partial \beta_0}\right) & E_{M_q}\left(\frac{\partial^2 \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_1^2}\right) \end{bmatrix}$$

$$B(\beta) = \begin{bmatrix} E_{M_q}\left[\left(\frac{\partial \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_0}\right)^2\right] & E_{M_q}\left(\frac{\partial \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_0} \cdot \frac{\partial \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_1}\right) \\ E_{M_q}\left(\frac{\partial \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_1} \cdot \frac{\partial \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_0}\right) & E_{M_q}\left[\left(\frac{\partial \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})}{\partial \beta_0}\right)^2\right] \end{bmatrix}$$

and

$$C(\beta) = [A(\beta)]^{-1} B(\beta) [A(\beta)]^{-1}$$

Asymptotic Distribution of MLEs

- If model assumption is correct (M_l=M_q) then -A(β) gives the usual Fisher information matrix.

- Fit model M_l to the data by MLE methods
- $\hat{\beta}$ denotes the MLE of β.
- By Theorem 3.2 of White (1982), $\sqrt{n}\,(\hat{\beta}- \beta^*)$ is asymptotically normal with mean **0** and variance covariance matrix C(β= β^*)

Criteria for Robust Test Planning

- Quantity of interest is

$$N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)=\ln\left(-\ln\left[R_{S_0}(\hat{\boldsymbol{\beta}},\varphi)\right]\right)=\frac{1}{\sigma}\left[\ln(\varphi)-\left(\hat{\beta}_0+\hat{\beta}_1 S_0\right)\right]$$

- We minimize the Asymptotic Mean Squared Error (AMSE) of the estimator above.

Criteria for Robust Test Planning

Asymptotic Variance is

$$n\,AVAR\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)\mid M_q\right]=\begin{bmatrix}\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)}{\partial\hat{\beta}_0}\\ \frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)}{\partial\hat{\beta}_1}\end{bmatrix}C(\boldsymbol{\alpha}:\boldsymbol{\beta}=\boldsymbol{\beta}^*)\begin{bmatrix}\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)}{\partial\hat{\beta}_0} & \frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)}{\partial\hat{\beta}_1}\end{bmatrix}$$

$$\text{where }\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)}{\partial\hat{\beta}_0}=-\frac{1}{\sigma}\text{ and }\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)}{\partial\hat{\beta}_1}=-\frac{S_0}{\sigma}.$$

Asymptotic Squared Bias is

$$\left(ABIAS\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)\mid M_q\right]\right)^2=\frac{1}{\sigma^2}\left[\mu_{q0}-(\beta_0^*+\beta_1^*S_0)\right]^2$$

We minimize the Asymptotic Mean Squared Error (AMSE):

$$AMSE\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)\mid M_q\right]=\left(ABIAS\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)\mid M_q\right]\right)^2+AVAR\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)\mid M_q\right]$$

Example

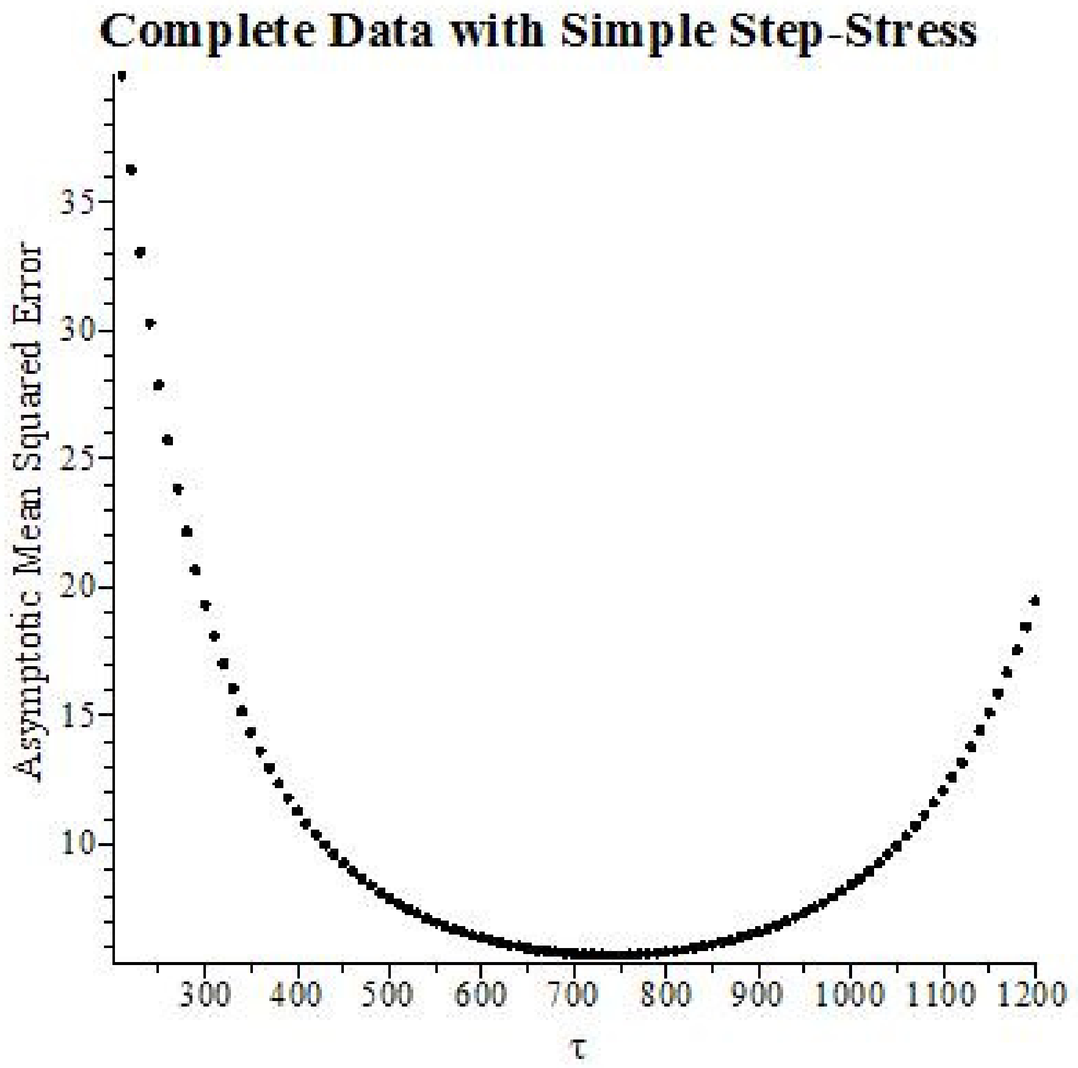
Revisit the same example used earlier.

Suppose that $ln(\theta_{qi})=\alpha_0+\alpha_1 S_i+\alpha_2 S_i^2$ is true with

- α_0=10.39264742
- α_1=-0.253190592
- α_2=0.004

The designs are derived by minimizing AMSE with respect to the stress-changing time.

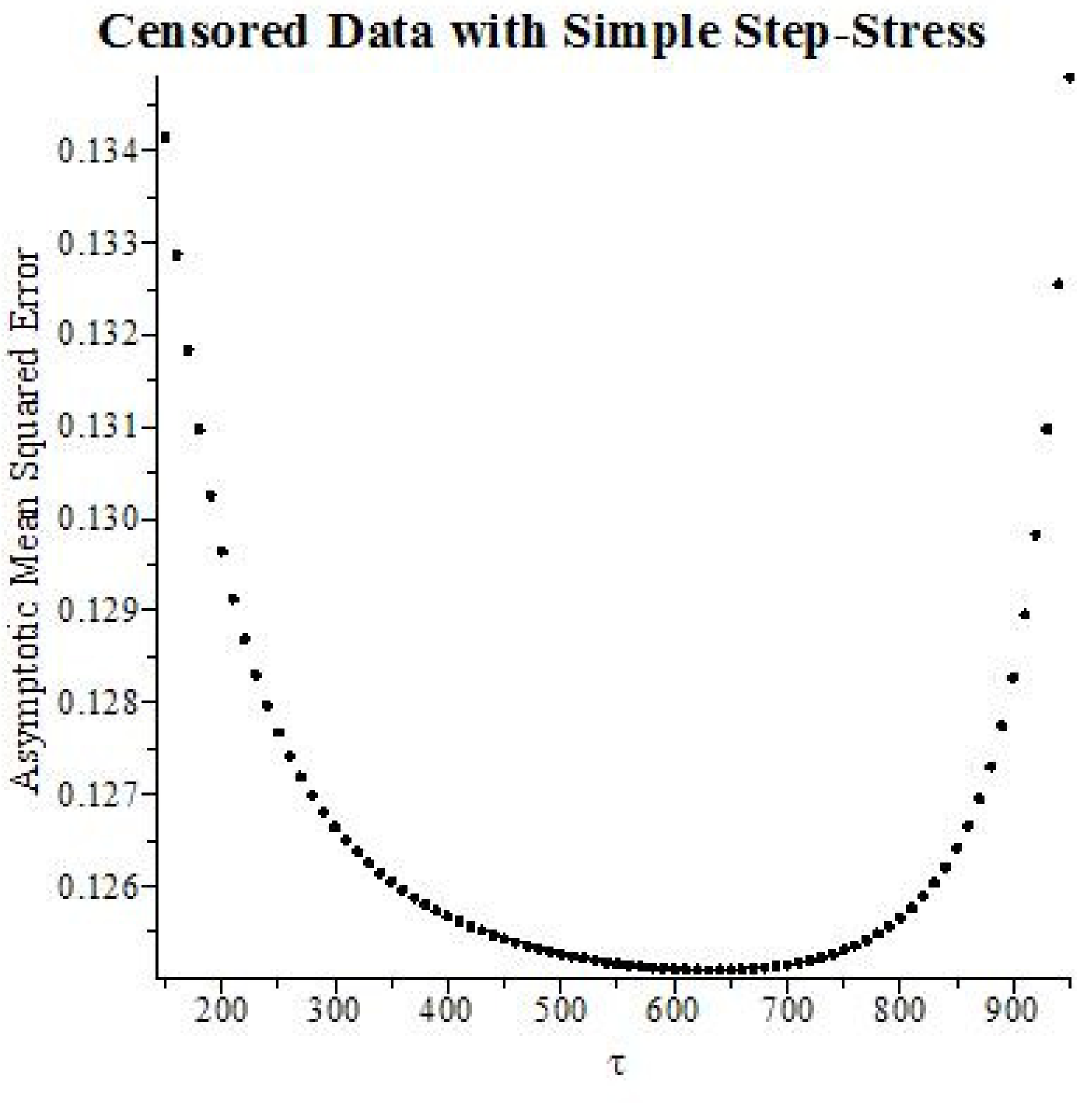
Minimizing AMSE (Complete)



The stress-changing time which minimizes the AMSE is at 739 minutes.

(optimal design is at 832)

Minimizing AMSE (Censored)



The stress-changing time which minimizes the AMSE is at 633 minutes.

(optimal design is at 756)

Future Study

- Current study is rather preliminary.

- Construct the optimal designs when the simultaneous estimation is needed for the parameters involved in both the scale-stress and shape-stress relationships.

- Investigate the optimal and robust designs for the multiple step stress model which minimizes the loss with respect to the stress-changing time(s) and middle stress level(s).

- Robust designs for simple and multiple step-stress ALT when a cumulative exposure model is used.

Reference

1. Bai, D.S., M.S. Kim, 1993. Optimum simple step-stress accelerated life tests for Weibull distribution and Type I censoring. *Naval Research Logistics*, 40, 193-210.
2. Bai, D.S., M.S. Kim, S.L. Kee, 1989. Optimum simple step-stress accelerated life tests with censoring. *IEEE Transactions on Reliability*, 38, 528-532.
3. Dharmadhikari, A.D., M. Rahman, 2002. Step-stress accelerated life testing for Weibull distribution with non-constant shape parameter. *Calcutta Statistical Association Bulletin*, 53, 33-51.
4. Ginebra, J., Sen, A., 1998. Minimax approach to accelerated life tests. *IEEE Transactions on Reliability*, 47, 261-267.
5. Khamis, I.H., 1997. Comparison between constant and step-stress tests for Weibull models. *International Journal of Quality and Reliability Management*, 14, 74-81.
6. Khamis, I.H., Higgins, S.H., 1998. A new model for step-stress testing. *IEEE Transactions on Reliability* , 47, 131–134.
7. Ma, H., W.Q. Meeker, 2008. Optimal step-stress accelerated life test plans for log-location-scale distributions. *Naval Research Logistics*, 55, 551-562.

Reliability, 32, 59-65.

Reference

8. Miller, R., W. Nelson, 1983. Optimum simple step-stress plans for accelerated life testing. *IEEE Transactions on Reliability*, *32, 59-65.*

9. W. B. Nelson,W.B., 1980. Accelerated life testing -- step-stress models and data analysis. *IEEE Transactions on Reliability*, *29, 103–108.*

10. Pascual, F. 2006. Accelerated Life Test Plans Robust to Misspecification of the Stress-Life Relationship. *Technometrics*, 48, 11-25.

11. Pascual, F., Montepiedra, G., 2003. Model-robust test plans with applications in accelerated life testing. *Technometrics*, 45, 47-57.

12. White, H. 1982. Maximum Likelihood Estimation of Mis-specified Models. Econometrica, 50, 1-25.

Contact information

Xiaojian Xu

Department of Mathematics
Brock University
St. Catharines, Ontario, Canada

Email: xxu@brocku.ca

Thank you for your attention!