Design of an Adaptive MCUSUM Control Chart
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Abstract
This method is based on the multivariate CUSUM control chart proposed by Pignatiello and Runger in 1990. We used the exponentially moving weighted average (EMWA) statistic to estimate the current process mean shift and change the reference value adaptively in each run. By specifying the minimal magnitude of the mean shift through the non-centrality parameter, our proposed control chart can achieve an overall good performance for detecting a range of shifts rather than a single value. We compared our adaptive multivariate CUSUM method with another adaptive multivariate as well as non-adaptive (conventional) control charts.

Introduction

We consider the multivariate sum
\[ C_t = \sum_{i=t-n+1}^{t} (X_i - \mu_0), \]
where \( n \) can be interpreted as the number of subgroups since the most recent renewal (i.e. zero values) of the CUSUM.
\[ MC_t = \max\{||C_t|| - kn_t, 0\} \]
where \( n_t = n_{t-1} + 1 \), if \( MC_{t-1} > 0 \); otherwise \( n_t = 1 \). The Pignatiello's MCUSUM scheme signals a shift in mean when \( MC_t > H_t \). \( H_t \) depends on the choice of reference value \( k \) and the in-control ARL0.

Pignatiello's MCUSUM charts

Motivation and Design of the A-MCUSUM
- Reference Value \( k \) determine the efficiency of MCUSUM, small \( k \) for small size shift, large \( k \) for large size shift.
- When detecting a Range of shift size, no unique \( k \) value. Propose the adaptive MCUSUM.
- To detect a range of shift size \([\lambda_{min},\lambda_{max}]\),

\[ C_t = \sum_{i=t-n+1}^{t} (X_i - \mu_0) \]

\[ ||C_t|| = \sqrt{\Sigma C_t'^2} \]

- we use \( y_t = \frac{||C_t||}{\max\{||C_t|| - kn_t, 0\}} \)
- We have \( k_t = \frac{2}{r} \) and \( n_t = n_{t-1} + 1 \), if \( MC_{t-1} > 0 \); otherwise \( n_t = 1 \). the process signals when \( y_t > H_t \), \( H_t \) is the control limit to maintain a desired ARL0.

Estimation of \( \lambda_t \)
- First, we use a vector-type EWMA statistic to cumulate the information from the sample readings, which is derived as
\[ Z_t = (1-r)Z_{t-1} + r(X_t - \mu_0) \]
where \( Z_0 = 0 \) and \( r \in (0,1) \) is a smoothing parameter.
- Then we use the quadratic form \( E(Z_t'Z_t) \) to get
\[ \lambda_t^2 = \frac{E(Z_t'Z_t)}{E(Z_t'Z_t)^2} \]
- Another EWMA operator is used to estimate the true mean shift \( \lambda_t^2 \) with the restriction of only detecting shifts larger than \( \lambda_{min} \), that is
\[ \lambda_t^2 = \max\{\lambda_{min}^2 (1 - r)\lambda_{min}^2 + r\lambda_t^2\} \]
The reason to be larger than \( \lambda_{min} \) is to increase the detection efficiency for the shifts larger than \( \lambda_{min} \).

Estimation of \( h(k) \)
- \( k_t = \frac{2}{r} \), we want to fit the model \( h(k, ARL_0) \) as
\[ h(k, ARL_0) = \exp(a(k) + \log(ARL_0)b(k)) \]
- when \( p = 2 \), the \( a(k) = 1.4659888 - 2.8142070k + 1.9093863k^2 - 0.5339259k^3 \)
and \( b(k) = 0.2095112 + 0.0479559k - 0.1288563k^2 + 0.0508699k^3 \)

The choice of smoothing parameter \( r \)
- The smoothing parameter \( r \) is only used in the multivariate EWMA statistic for the estimation of current shift, the \( \lambda_{min} = 0.5 \) and \( \lambda_0 = \lambda_{min} = 0.5 \)

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</table>

Guidelines for A-MCUSUM
- Set up the in-control average run length ARL0 in the process, select the detecting range of shifts \([\lambda_{min},\lambda_{max}]\).
- Choose the smoothing parameter \( 0.05 < r < 0.30 \)
- Choose the initial value, we may use the mean value of the \( \lambda_{min} \), and the \( \lambda_{max} \) as a good choice
- Select an appropriate control limit \( H_t \) to achieve desired in-control ARL0.
- We signal the process when \( y_t > H_t \).

Comparison with other control charts
- we will compare the adaptive Pignatiello CUSUM in this work with the non-adaptive Pignatiellos CUSUM, Crosiers CUSUM and the adaptive Crosiers CUSUM proposed in (Dai 2010).
- Detection large range shifts (0.5, 4.0).
- \( r = 0.20 \) for Adaptive Crosiers and Adaptive Pignatiellos CUSUM
- \( \lambda_0 = \lambda_{min} = \lambda_{max} \) for the non-adaptive CUSUM (Crosiers and Pignatiello) , we use different reference \( k = \lambda_{min}, \lambda_{max} \) separately. MC1 and MC2 are adaptive Pignatiellos' CUSUM and adaptive Crosier's CUSUM; MC1 - MC4 are Pignatiellos' CUSUM (\( k = 0.25 \)), Pignatiellos' CUSUM (\( k = 2 \)), Crosier's CUSUM (\( k = 0.25 \)), Crosier's CUSUM (\( k = 2 \)) separately.

Conclusions
- The Shewhart Control Chart can only be good to detect the large shifts
- The EWMA control chart and the CUSUM control charts are good to detect the small and moderate shifts when choose the smaller reference value ??.
- The adaptive Crosiers CUSUM and the adaptive Pignatiellos CUSUM, they solve the problems when we want to detect a range of shift size which including large shifts and small shifts;
- The Adaptive Pignatiellos CUSUM we proposed performs almost uniformly better than the adaptive Crosiers CUSUM. Of course much better than the other non-adaptive CUSUMs.

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