Design of an Adaptive MCUSUM Control Chart

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Abstract

This method is based on the multivariate CUSUM control chart proposed by Pignatiello and Runger in 1990. We used the exponentially moving weighted average (EMWA) statistic to estimate the current process mean shift and change the reference value adaptively in each run. By specifying the minimal magnitude of the mean shift through the non-centrality parameter, our proposed control chart can achieve an overall good performance for detecting a range of shifts rather than a single value. We compared our adaptive multivariate CUSUM method with another adaptive multivariate as well as non-adaptive (conventional) control charts.

Introduction

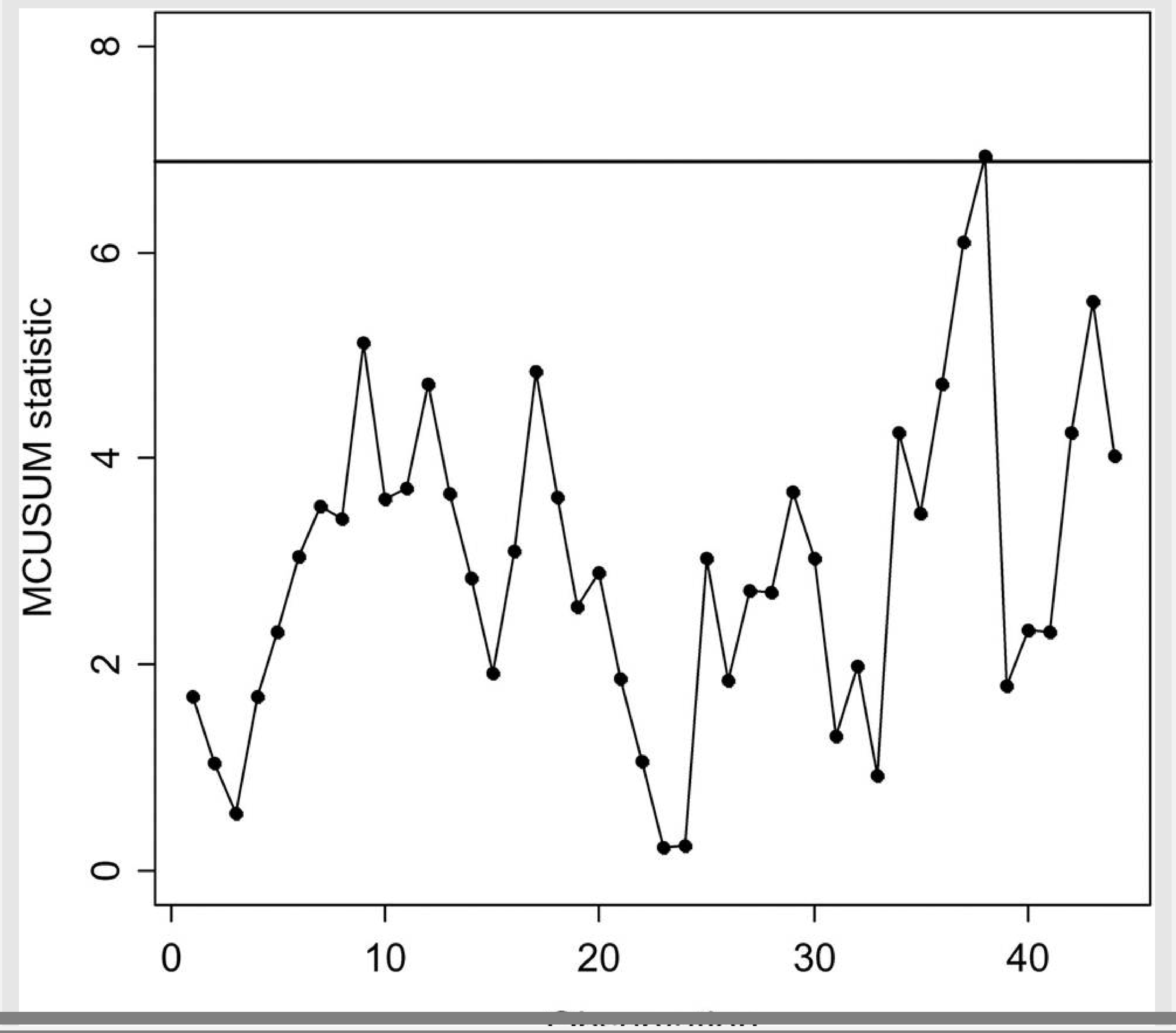
We consider the multivariate sum $\mathbf{C_t} = \sum_{i=t-n_t+1}^{t}(\mathbf{X_i} - \mu_0)$, where $\mathbf{n_t}$ can be interpreted as the number of subgroups since the most recent renewal (i.e. zero values) of the CUSUM.

$$MC_t = \max\{||C_t|| - kn_t, 0\}$$

where

$\mathbf{n_t = n_{t-1} + 1}$, if $\mathbf{MC_{t-1} > 0}$; otherwise $\mathbf{n_t = 1}$
The Pignatiellos MCUSUM scheme signals a shift in mean when $\mathbf{MC_t > H_c}$. $\mathbf{H_c}$ depends on the choice of reference value $\mathbf{k}$ and the in-control $\mathbf{ARL_0}$

Pignatiellos MCUSUM charts



Motivation and Design of the A-MCUSUM

- Reference Value **k** determine the efficiency of MCUSUM, small **k** for small size shift, large **k** for large size shift.
- When detecting a Range of shift size, no unique **k** value. Propose the adaptive MCUSUM.
- To detect a range of shift size $[\lambda_{min}, \lambda_{max}]$;

$$C_t = \sum_{i=t-n_t+1}^{t}(X_i - \mu_0)$$

$$||C_t|| = \sqrt{C_t'\Sigma^{-1}C_t}$$

- we ust $\mathbf{y_t} = \frac{\max\{||C_t|| - k_t n_t, 0\}}{h(k_t)}$
- We have $\mathbf{k_t} = \frac{\lambda_t}{2}$ and $\mathbf{n_t = n_{t-1} + 1}$, if $\mathbf{MC_{t-1} > 0}$; otherwise $\mathbf{n_t = 1}$; the process signals when $\mathbf{y_t > H}$, where $\mathbf{H}$ is the control limit to maintain a described $\mathbf{ARL_0}$ and its value is close to 1.

Estimation of λ_t

- First, we use a vector-type EWMA statistic to cumulate the information from the sample readings, which is derived as

$$Z_t = (1-r)Z_{t-1} + r(X_t - \mu_0)$$

Where $\mathbf{Z_0 = 0}$ and $\mathbf{r} \in (0, 1)$is a smoothing parameter

- Then we use the quadratic form $\mathbf{E(Z_t'\Sigma^{-1}Z_t)}$ to get

$$\lambda_t^2 = \frac{\{Z_t'\Sigma^{-1}Z_t - [1-(1-r)^{2t}]\frac{rp}{2-r}\}}{[1-(1-r)^t]^2}$$

- Another EWMA operator is used to estimate the true mean shift λ_t^2 with the restriction of only detecting shifts larger than λ_{min}, that is

$$\hat{\lambda_t^2} = \text{Max}\{\lambda_{min}^2, (1-r)\hat{\lambda_{t-1}^2} + r\lambda_t^2\}$$

The reason to be larger than λ_{min} is to increase the detection efficiency for the shifts larger than λ_{min}.

Estimation of $h(k_t)$

- $\mathbf{k_t} = \frac{\lambda_t}{2}$, we want to fit the model $\mathbf{h(k, ARL_0)}$ as

$$h(k, ARL_0) = \text{Exp}(a(k) + \text{Log}(ARL_0)b(k))$$

- when $\mathbf{p = 2}$, the $\mathbf{a(k) = 1.4659888 - 2.8142070k + 1.9093863k^2 - 0.53392594k^3}$ and $\mathbf{b(k) = 0.2095112 + 0.0479559k - 0.1288563k^2 + 0.05086997k^3}$

The choice of smoothing parameter r

- The smoothing parameter **r** is only used in the multivariate EWMA statistic for the estimation of current shift, the $\lambda_{min} = 0.5$ and $\hat{\lambda_0} = \lambda_{min} = 0.5$

Shift	r=0.05	r=0.10	r=0.20	r=0.30	r=0.50
0.00	200.26	202.18	200.49	201.86	202.92
0.50	25.39	25.47	25.58	26.90	30.54
1.00	9.71	9.45	9.07	9.06	9.40
1.50	5.66	5.35	5.01	4.89	4.80
2.00	3.93	3.65	3.35	3.22	3.08
2.50	3.02	2.78	2.50	2.38	2.24
3.00	2.46	2.23	2.00	1.88	1.77
3.50	2.10	1.90	1.67	1.56	1.45
4.00	1.83	1.63	1.42	1.33	1.25

$\hat{\lambda_0}$ choice

- The range of shift size is from 0.50 to 4, the smoothing parameter $\mathbf{r = 0.20}$.
- we have different choices of $\hat{\lambda_0} = 0.00, 0.50, 1.375, 2.25, 3.125, 4.00$

Shift	0.00	0.50	1.375	2.25	3.125	4.00
0.00	201.86	202.70	199.99	200.09	199.04	199.21
0.50	25.60	25.53	25.17	25.03	25.86	27.29
1.00	9.15	9.07	8.83	8.73	9.31	10.27
1.50	5.02	5.03	4.78	4.70	4.98	5.63
2.00	3.39	3.36	3.17	3.03	3.15	3.43
2.50	2.53	2.51	2.35	2.21	2.22	2.36
3.00	2.03	2.00	1.87	1.74	1.71	1.75
3.50	1.70	1.68	1.55	1.43	1.39	1.40
4.00	1.43	1.42	1.31	1.24	1.20	1.19

Guidelines for A-MCUSUM

- Set up the in-control average run length $\mathbf{ARL_0}$ in the process, select the detecting range of shifts $[\lambda_{min}, \lambda_{max}]$
- Choose the smoothing parameter $\mathbf{0.05 < r < 0.30}$
- Choose the initial value, we may use the mean value of the λ_{min} and the λ_{max} as a good choice
- Select an appropriate control limit **H** to achieve desired in-control $\mathbf{ARL_0}$
- We signal the process when $\mathbf{y_t > h}$.

Comparison with other control charts

- we will compare the adaptive Pignatiellos CUSUM in this work with the non-adaptive Pignatiellos CUSUM, Crosiers MCUSUM and the adaptive Crosiers CUSUM proposed in (Dai 2010).
- Detection large range shifts (0.5, 4.0)
- $\mathbf{r = 0.20}$ for Adaptive Crosiers and Adaptive Pignatellos CUSUM
- $\hat{\lambda_0} = \frac{\lambda_{min}+\lambda_{max}}{2}$; for the non-adaptive MCUSUM (Crosiers and Pignatellos) , we use different reference $\mathbf{k} = \lambda_{min}, \lambda_{max}$separately. **MC1** and **MC2** are adaptive Pignatiello's MCUSUM and adaptive Crosier's MCUSUM; **MC1** − **MC4** are Pignatiello's MCUSUM ($\mathbf{k = 0.25}$), Pignatiello's MCUSUM ($\mathbf{k = 2}$), Crosier's MCUSUM ($\mathbf{k = 0.25}$), Crosier's MCUSUM ($\mathbf{k = 2}$)seperately.

Shift	AMC1	MC1	MC2	AMC2	MC3	MC4
0.00	200.09	202.16	202.48	200.00	200.00	200.00
0.50	25.03	25.71	110.36	30.45	26.50	99.86
1.00	8.73	10.31	35.02	11.56	11.44	30.36
1.50	4.70	6.50	11.72	5.75	7.30	10.32
2.00	3.03	4.80	4.95	3.55	5.41	4.57
2.50	2.21	3.85	2.74	2.52	4.33	2.64
3.00	1.74	3.25	1.86	1.96	3.63	1.84
3.50	1.43	2.01	1.43	1.61	3.16	1.43
4.00	1.24	1.83	1.19	1.37	2.82	1.21

Conclusions

- The Shewhart Control Chart can only be good to detect the large shifts
- The EWMA control chart and the CUSUM control charts are good to detect the small and moderate shifts when choose the smaller reference value ??.
- The adaptive Crosiers CUSUM and the adaptive Pignatellos CUSUM, they solve the problems when we want to detect a range of shift size which including large shifts and small shifts;
- The Adaptive Pignatiellos CUSUM we proposed performs almost uniformly better than the adaptive Crosiers CUSUM. Of course much better than the other non-adaptive CUSUMs.

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