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Optimal Designs for Prediction of Individual Effects in RCR Models

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Abstract: We propose optimal designs for the prediction of the individual response in random coefficient regression models with unknown population parameters for finite numbers of individuals.

Random Coefficient Regression

$$\boxed{Y_{ij} = \mathbf{f}(x_{ij})^{\mathsf{T}}\boldsymbol{\beta}_i + \varepsilon_{ij}} \qquad x_{ij} \in \chi,\ j = 1, \dots, m_i,\ i = 1, \dots, n$$

$\mathrm{E}(\varepsilon_{ij}) = 0,\ \mathrm{Var}(\varepsilon_{ij}) = \sigma^2$

$\mathrm{E}(\boldsymbol{\beta}_i) = \boldsymbol{\beta},\ \ \mathrm{Cov}(\boldsymbol{\beta}_i) = \sigma^2\mathbf{D}$

uncorrelated

Equal individual settings $\quad m_i = m \ \ \&\ \ x_{ij} = x_j$

$$\boxed{\mathbf{Y}_i = \mathbf{F}\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i}$$

$$\mathbf{Y}_i = (\mathbf{Y}_{i1}, \dots, \mathbf{Y}_{im})^{\mathsf{T}}, \quad \mathbf{F} = (\mathbf{f}(x_1), \dots, \mathbf{f}(x_m))^{\mathsf{T}}$$

Best Linear Unbiased Predictor for individual parameters

$$\boxed{\hat{\boldsymbol{\beta}}_i = \mathbf{D}\left((\mathbf{F}^{\mathsf{T}}\mathbf{F})^{-1} + \mathbf{D}\right)^{-1}\hat{\boldsymbol{\beta}}_{i,\mathrm{ind}} + (\mathbf{F}^{\mathsf{T}}\mathbf{F})^{-1}\left((\mathbf{F}^{\mathsf{T}}\mathbf{F})^{-1} + \mathbf{D}\right)^{-1}\hat{\boldsymbol{\beta}}}$$

$$\boxed{\hat{\boldsymbol{\beta}} = (\mathbf{F}^{\mathsf{T}}\mathbf{F})^{-1}\mathbf{F}^{\mathsf{T}}\overline{\mathbf{Y}} \quad \& \quad \hat{\boldsymbol{\beta}}_{i,\mathrm{ind}} = (\mathbf{F}^{\mathsf{T}}\mathbf{F})^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{Y}_i} \qquad \overline{\mathbf{Y}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i$$

Mean Squared Error matrix for prediction

$$\boxed{(\mathbf{I}_n - \tfrac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathsf{T}}) \otimes (\mathbf{F}^{\mathsf{T}}\mathbf{F} + \mathbf{D}^{-1})^{-1} + \tfrac{1}{n}(\mathbf{1}_n\mathbf{1}_n^{\mathsf{T}}) \otimes (\mathbf{F}^{\mathsf{T}}\mathbf{F})^{-1}}$$

Optimal design

distinct settings $\quad x_1, \dots, x_k$

numbers of replications $\quad m_1, \dots, m_k \qquad \sum_{j=1}^{k} m_j = m$

Individual design

$$\xi = \begin{pmatrix} x_1, & \dots, & x_k \\ m_1, & \dots, & m_k \end{pmatrix}$$

Information (fixed effects model)

$$\mathbf{M}(\xi) = \sum_{j=1}^{k} m_j \mathbf{f}(x_j)\mathbf{f}(x_j)^{\mathsf{T}}$$

Integrated Mean Squared Error criterion

$$\boxed{\Phi(\xi) = \mathrm{tr}\left(\mathbf{M}(\xi)^{-1}\mathbf{V}\right) + (n-1)\,\mathrm{tr}\left(\left(\mathbf{M}(\xi) + \mathbf{D}^{-1}\right)^{-1}\mathbf{V}\right)}$$

$$\mathbf{V} = \int_{\chi} \mathbf{f}(x)\mathbf{f}(x)^{\mathsf{T}}\nu(\mathrm{d}x)$$

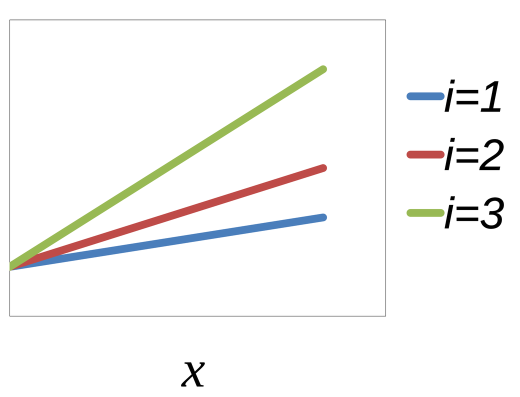
***weighted sum** of IMSE-criterion in fixed effects models and Bayesian IMSE-Criterion* **!**

Optimality can be checked for approximate designs by means of the general equivalence theorem

Example: *straight line regression*

$$Y_{ij} = \boldsymbol{\beta}_{i1} + \boldsymbol{\beta}_{i2}x_j + \varepsilon_{ij}, \quad x_{ij} \in [0, 1]$$

$\mu_i(x) = \boldsymbol{\beta}_{i1} + \boldsymbol{\beta}_{i2}x$



$$\mathrm{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ n = 100,\ m = 10,\ d_1 = 0.01$$

Individual design $\quad \xi = \begin{pmatrix} 0 & 1 \\ m - m_1 & m_1 \end{pmatrix}$

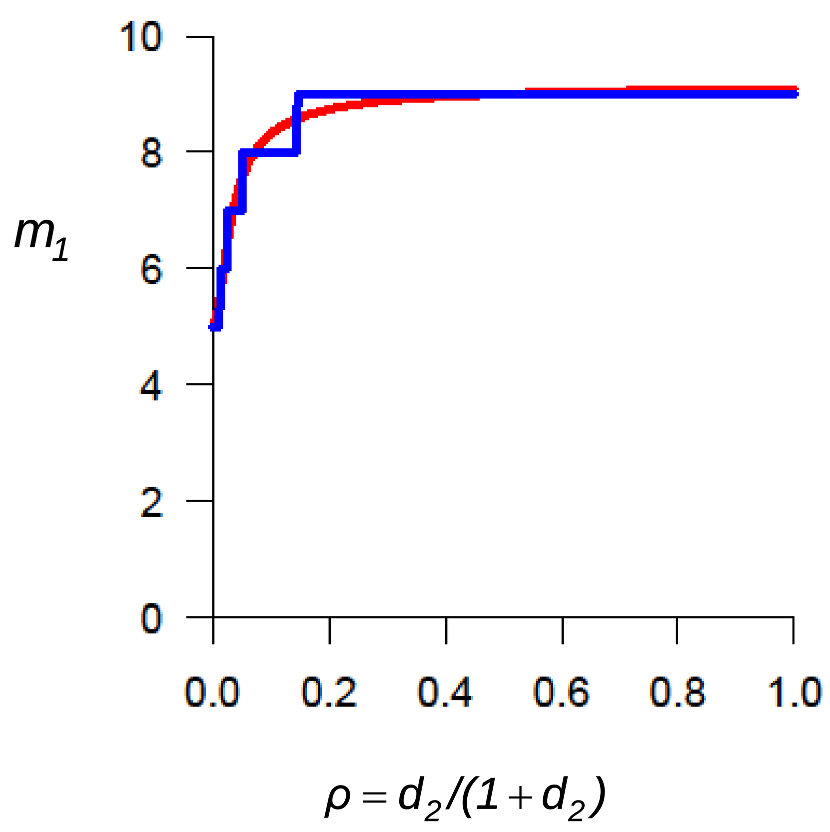


Fig.1: Optimal frequency (—) and the frequency assigned by the exact design (—) in dependence on the variance parameter ρ

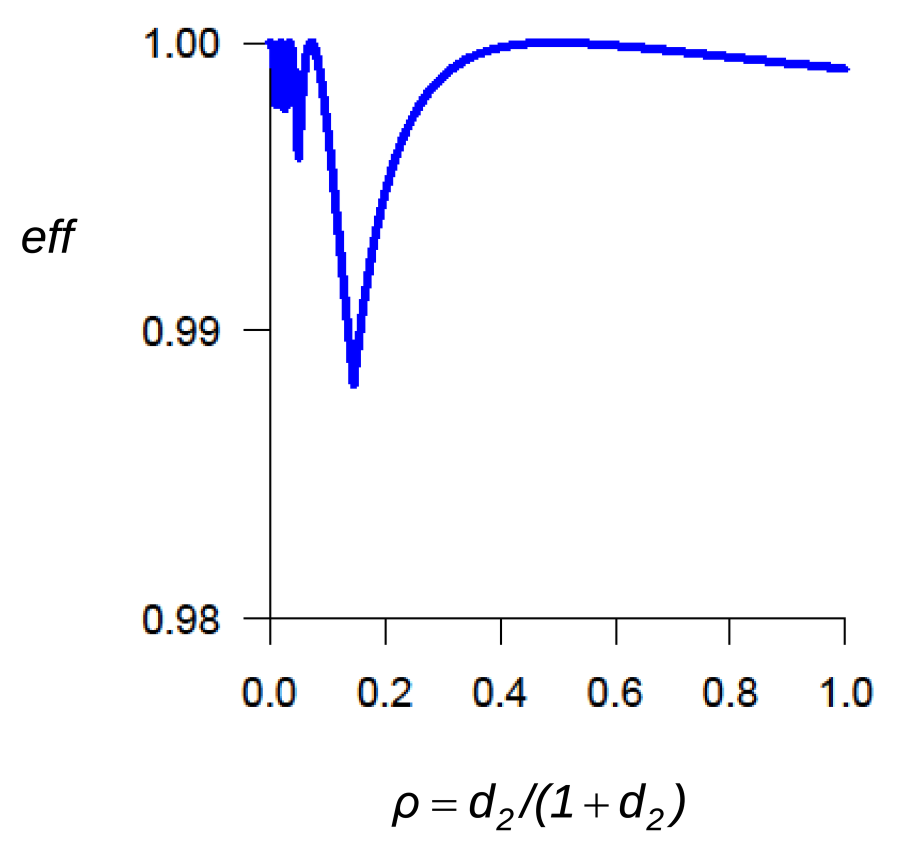


Fig.2: Efficiency of the exact design in dependence on the variance parameter ρ

Literature: Prus, M., Schwabe, R.: Optimal Designs for Individual Prediction in Random Coefficient Regression Models. Proceedings of the ODE,Vienna 2011

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