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Tractable Modeling of Systems with Functional Outputs with Nonseparable Covariance Structures

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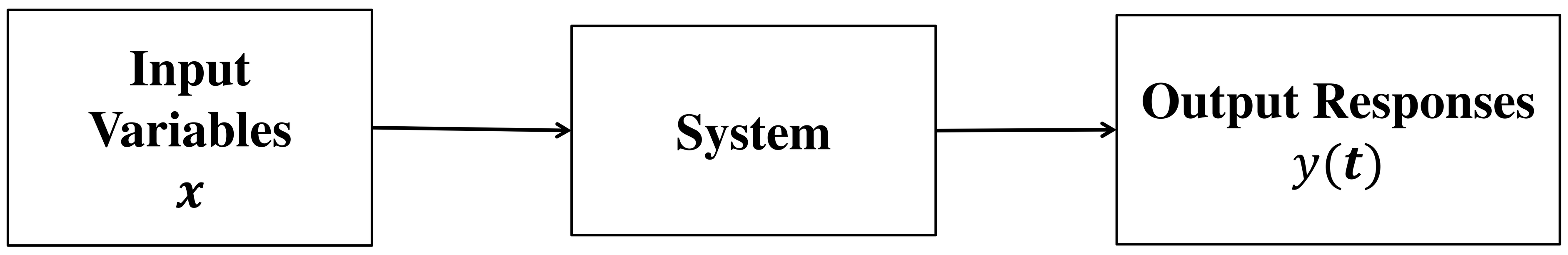
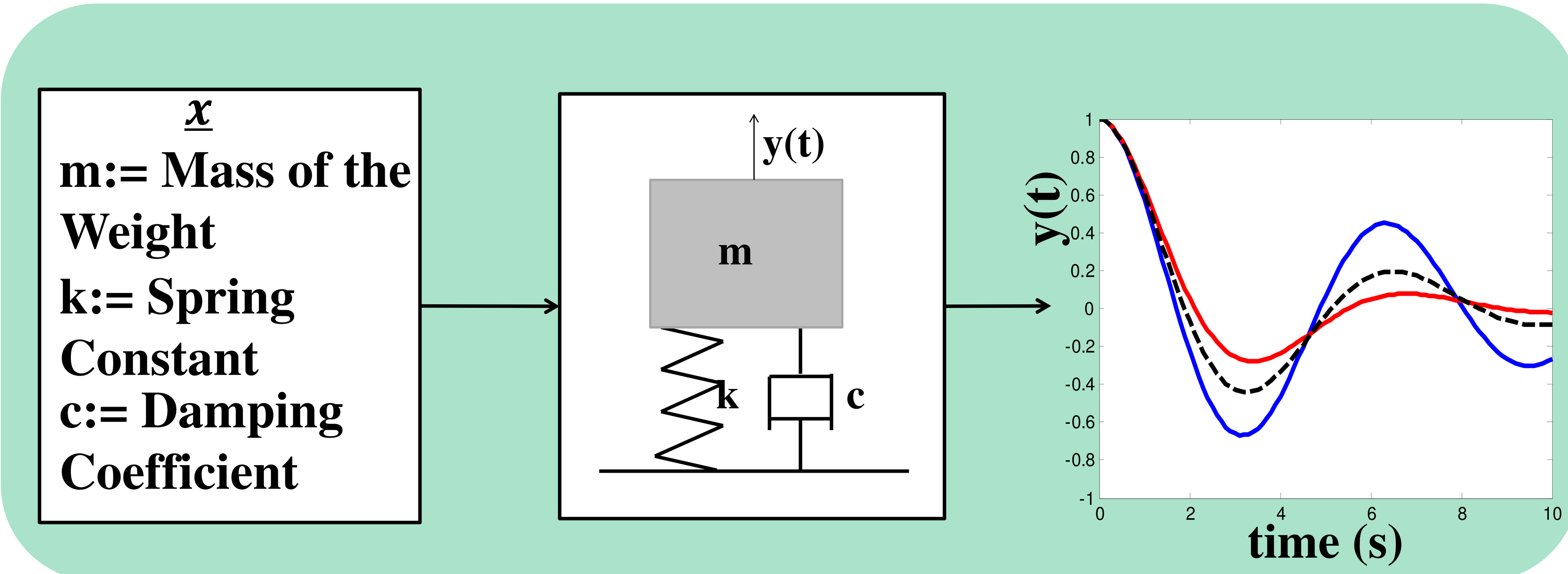
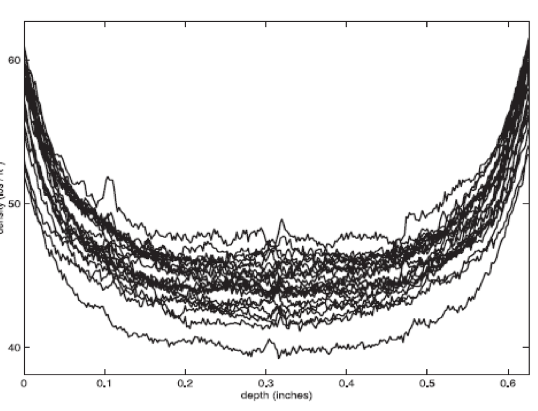
Motivation

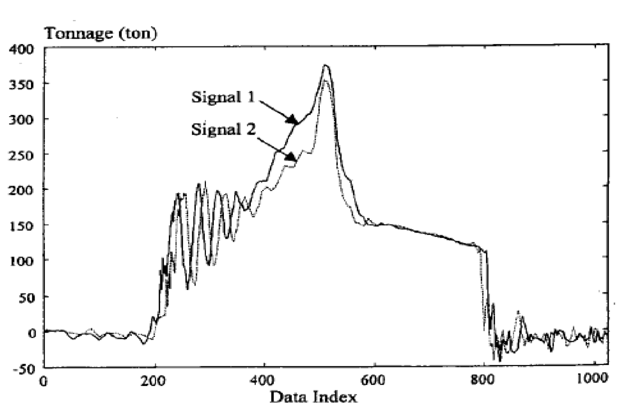




Examples of Functional Data Responses



Particle Board Density.
P. M. Winistorfer, *et al.(1996)*



Stamping Tonnage Signals.
Jin and Shi(1999)

Various etching profiles from Deep reactive-ion etching Process.
C. Zou, et al.(2007)

Objective

To predict a new functional response using previously observed functional responses.

Introduction

Gaussian Process Model

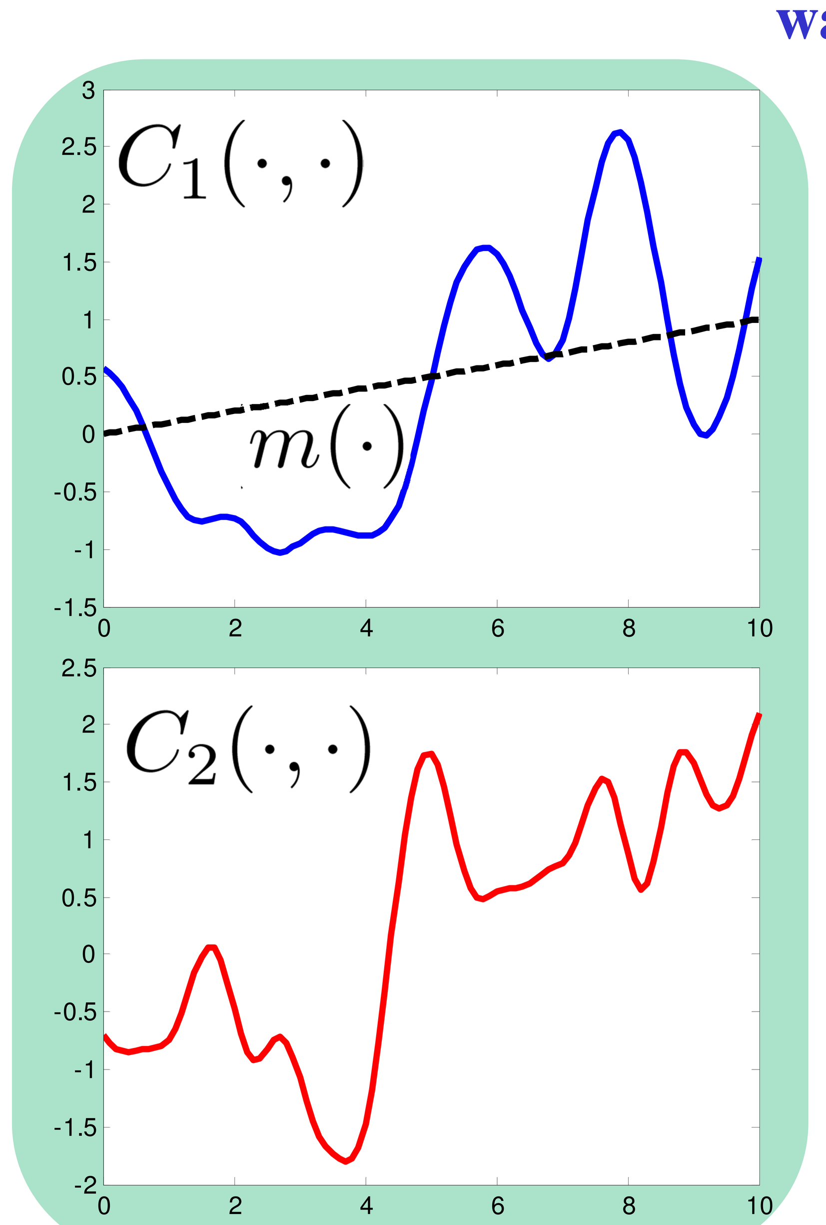
A Gaussian process (GP) is a probability distribution for an unknown function

- Colloquially, an infinite dimensional multivariate normal distribution

$$Z(x) \sim GP[m(\cdot), C(\cdot,\cdot)]$$

m(.) is the mean function

C(.,.) is the covariance function



GP model for functional responses

$$y(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{f}^{\mathsf{T}}(\boldsymbol{x},\boldsymbol{t})\boldsymbol{\beta} + z(\boldsymbol{x},\boldsymbol{t})$$

We observe responses:

$$\boldsymbol{Y} = [\boldsymbol{y}_1^{\mathsf{T}}, \dots, \boldsymbol{y}_m^{\mathsf{T}}]^{\mathsf{T}}$$

Prediction

1. Estimate the parameters using the likelihood (Maximum likelihood, Bayesian, etc.):

$$l \propto exp\left[-\frac{1}{2}\left\{log\,|\boldsymbol{\Sigma}| + (\boldsymbol{Y}-\boldsymbol{F\beta})^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y}-\boldsymbol{F\beta})\right\}\right]$$

$\boldsymbol{\Sigma}$ is $nm \times nm$
operations: $\mathcal{O}\left(n^3m^3\right)$
bytes of storage: $\mathcal{O}\left(n^2m^2\right)$

2. Develop the best linear unbiased predictor to predict new responses:

$$\hat{y}\left(\boldsymbol{x}_0,\boldsymbol{t}_0\right) = \boldsymbol{f}^{\mathsf{T}}(\boldsymbol{x}_0,\boldsymbol{t}_0)\boldsymbol{\beta} + \boldsymbol{\sigma}^{\mathsf{T}}\left(\boldsymbol{x}_0,\boldsymbol{t}_0\right)\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{Y}-\boldsymbol{F\beta}\right),$$

Previous Approach

Separable Covariance

$$C\left((\boldsymbol{x},\boldsymbol{t}),(\boldsymbol{x}',\boldsymbol{t}')\right) = C_x\left(\boldsymbol{x},\boldsymbol{x}'\right)C_t\left(\boldsymbol{t},\boldsymbol{t}'\right)$$

Used in (recent examples): Williams et al. (2006), Genton (2007), Rougier (2008), Bayarri et al. (2009), and Conti and O'Hagan (2010)

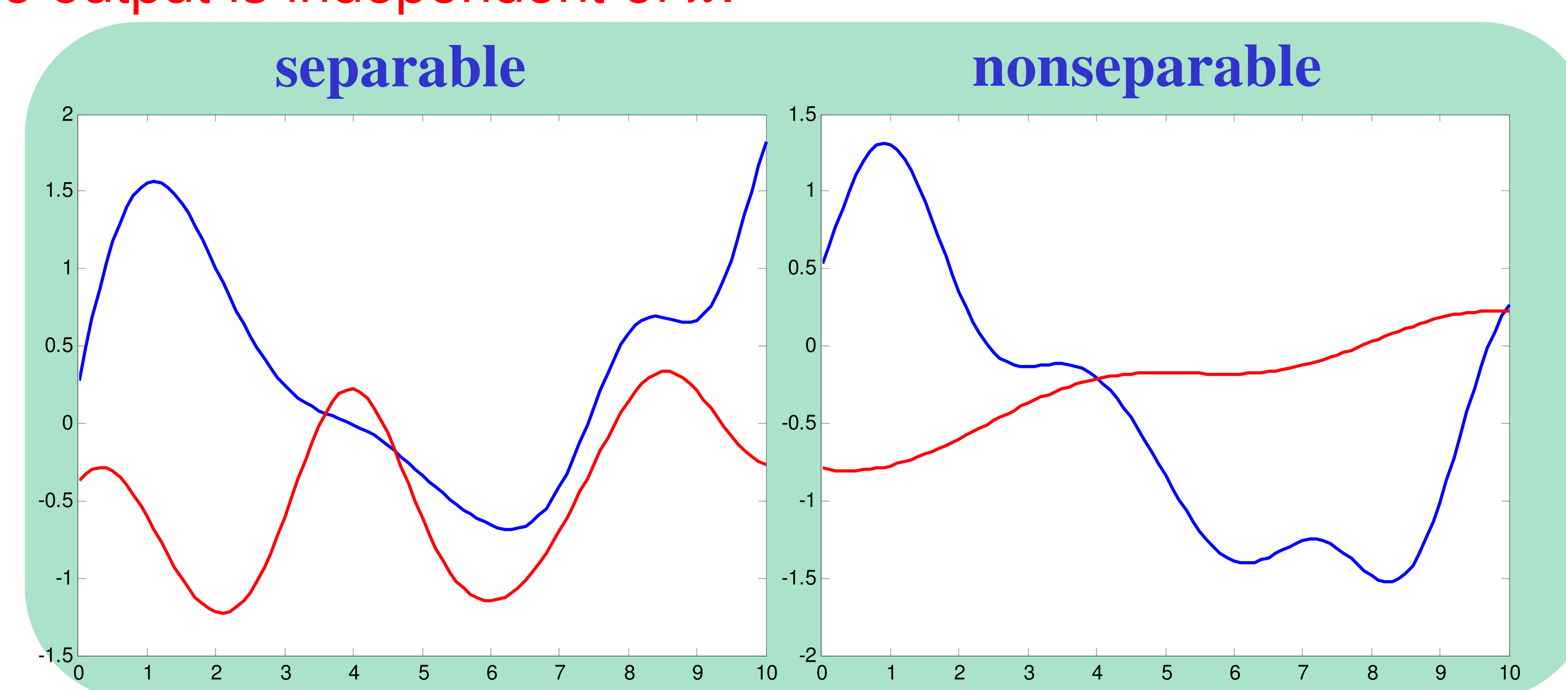
Under this assumption:

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_t \rightarrow \boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_x^{-1} \otimes \boldsymbol{\Sigma}_t^{-1}$$

Likelihood evaluation and prediction are much faster.

$\boldsymbol{\Sigma}_t$ is $n \times n$ and $\boldsymbol{\Sigma}_x$ is $m \times m$
operations: $\mathcal{O}\left(\max(n^3,m^3)\right)$
bytes of storage: $\mathcal{O}\left(\max(n^2,m^2)\right)$

However, Since $C_t\left(\boldsymbol{t},\boldsymbol{t}'\right)$ is independent of x, the covariance of the output is independent of x.



Accomplishments

We propose a nonstationary and nonseparable covariance function to model a wide variety of functional responses.

$$C\left((\boldsymbol{x},\boldsymbol{t}),(\boldsymbol{x}',\boldsymbol{t}')\right) = C_x\left(\boldsymbol{x},\boldsymbol{x}'\right)C_{t|x}\left(\boldsymbol{t},\boldsymbol{t}'|\boldsymbol{x},\boldsymbol{x}'\right)$$

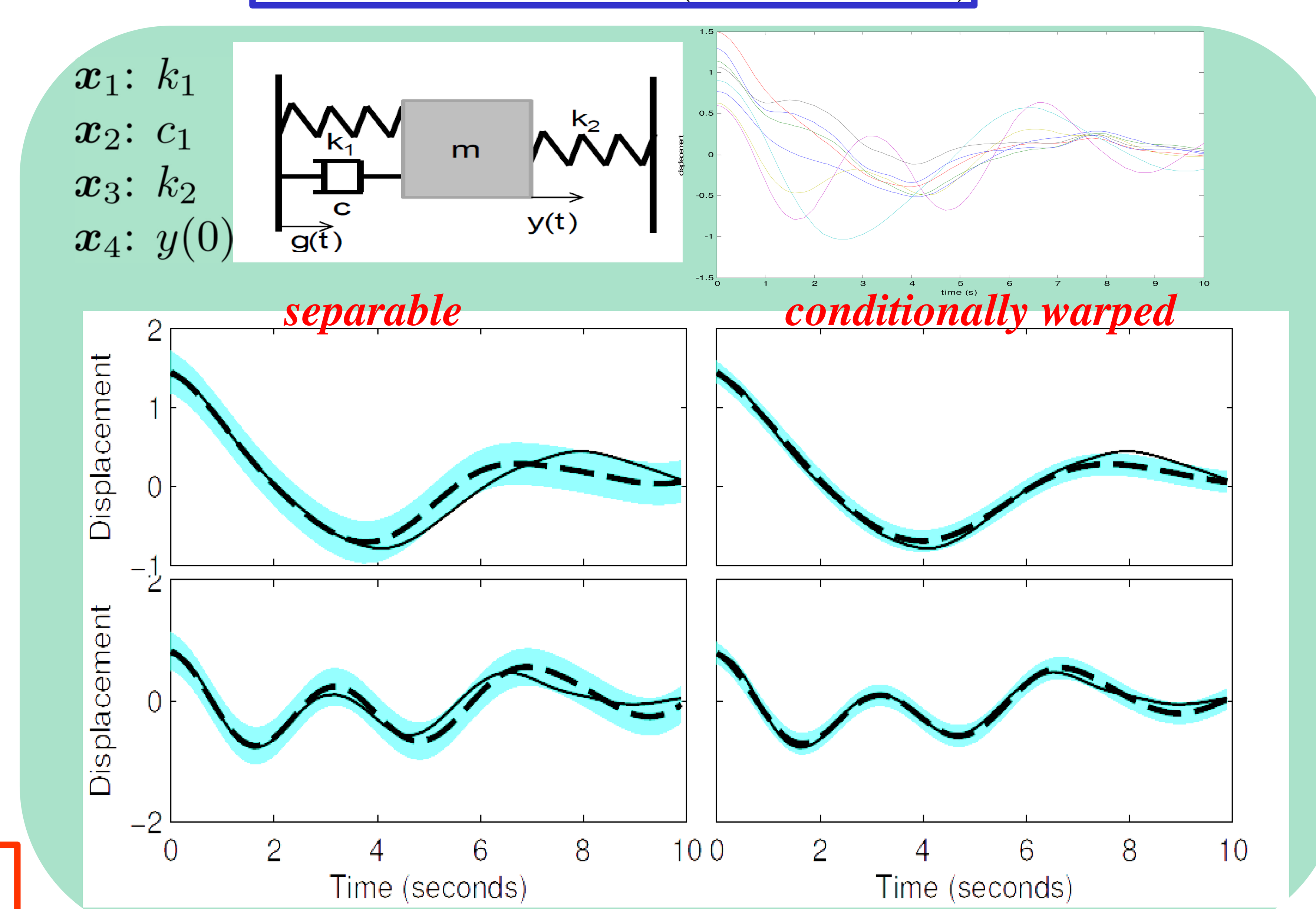
conditionally warped

$$C_{t|x}\left(\boldsymbol{t},\boldsymbol{t}'|\boldsymbol{x},\boldsymbol{x}'\right) = \boldsymbol{\sigma}_t^{\mathsf{T}}\left(\boldsymbol{t}|\boldsymbol{x}\right)\boldsymbol{\Sigma}_t^{-\frac{1}{2}}\left(\boldsymbol{x}\right)\boldsymbol{\Sigma}_t^{-\frac{1}{2}}\left(\boldsymbol{x}'\right)\boldsymbol{\sigma}_t\left(\boldsymbol{t}'|\boldsymbol{x}'\right)$$

where, $\boldsymbol{\sigma}_t^{\mathsf{T}}\left(\boldsymbol{t}\mid\boldsymbol{x}\right) = \left[C_t\left(\boldsymbol{t},\boldsymbol{t}_1\mid\boldsymbol{x}\right), C_t\left(\boldsymbol{t},\boldsymbol{t}_2\mid\boldsymbol{x}\right),\dots,C_t\left(\boldsymbol{t},\boldsymbol{t}_n\mid\boldsymbol{x}\right)\right]$
and $\left\{\boldsymbol{\Sigma}_t\left(\boldsymbol{x}\right)\right\}_{ij} = C_t\left(\boldsymbol{t}_i,\boldsymbol{t}_j\mid\boldsymbol{x}\right)$

Using this model, inference is still tractable.

operations: $\mathcal{O}\left(\max(mn^3,m^3)\right)$
bytes of storage: $\mathcal{O}\left(\max(n^2,m^2)\right)$



References

The associated paper contains references used in this poster.

Plumlee, M and Joseph, V. R. (2012). Tractable functional Response Modeling with Nonseparable and Nonstationary Covaraince. *Submitted*