Optimal Experimental Design for Systems with Bivariate Failures under a Bivariate Weibull Function

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Abstract

In manufacturing industry, it may be important to study the relationship between machine component failures under stress. Examples include failures such as integrated circuits and memory chips in electronic merchandise given levels of electronic shock. Such studies are important for the development of new products and for the improvement of existing products. We assume two component failures for simplicity and assume the probability of failures increases as a cumulative bivariate Weibull function with stress. Optimal designs have been developed for two correlated binary responses using the Gumbel model, the bivariate binary Cox model and the bivariate probit model. In all these models, the amounts of damage (failure) on two component range from negative to positive infinity.

In the Weibull model, the amounts of damage are positive which is natural for experimental factors such as voltage, tension or pressure.

In this paper, we discuss optimal designs under the bivariate Weibull assumptions.

Notation

A design is denoted

$$\xi = \{x_i, w_i\}_1^K,$$

where

1. K is the total number of design points in ξ,
2. x_i denotes the design point,
3. n_i denotes the sample size tested at the design point x_i,
4. $n = \sum_{i=1}^K n_i$,
5. $w_i = n_i/n$ denotes the weight at the design point x_i.

Two Dependent Binary Outcomes

1. Let U and V denote the outcomes for the first component and the second component, respectively, with 0 for no failure and 1 for failure.
2. Let $p_{uv}(x) = Pr(U = u, V = v | X = x), \{u, v\} \in \{0, 1\}$ denote the outcome probabilities given a stress x.

		1st Component		
		0 (Success)	1 (Failure)	
2nd Component	0 (Success)	p_{00}	p_{10}	$p_{\cdot 0}$
	1 (Failure)	p_{01}	p_{11}	$p_{\cdot 1}$
		$p_{0\cdot}$	$p_{1\cdot}$	1

Outcome Probabilities

Likelihood function

The likelihood function of a single system with two component failures U and V at stress x is

$$L(\theta|u, v; x) = p_{11}^{uv} \times p_{10}^{u(1-v)} \times p_{01}^{(1-u)v} \times p_{00}^{(1-u)(1-v)}.$$

Model

1. Let Y and Z denote amounts of damage on component 1 and component 2.
2. $f(y, z|x)$ is a bivariate Weibull regression model.
3. In the bivariate Weibull model, the domain of random variables are positive which is natural for experiment.

Dichotomization

Failures (damages) are defined by dichotomizing Y and Z:

$$U = \begin{cases} 0 \text{ (No fatal damage for the component 1)}, & \text{if } Y < v_1^* \\ 1 \text{ (Fatal damage for the component 1)}, & \text{otherwise}, \end{cases}$$

$$V = \begin{cases} 0 \text{ (No fatal damage for the component 2)}, & \text{if } Z < v_2^* \\ 1 \text{ (Fatal damage for the component 2)}, & \text{otherwise}, \end{cases}$$

where v_1^* and v_2^* are predetermined cutoff values.

Outcome Probabilities

$$p_{00} = \int_0^{v_1^*}\int_0^{v_2^*} f(y, z)dydz$$
$$p_{01} = \int_0^{v_1^*}\int_{v_2^*}^{\infty} f(y, z)dzdy$$
$$p_{10} = \int_{v_1^*}^{\infty}\int_0^{v_2^*} f(y, z)dzdy$$
$$p_{11} = \int_{v_1^*}^{\infty}\int_{v_2^*}^{\infty} f(y, z)dydz.$$

Locally Optimal Design

$$\xi^* = \arg\min_{\xi \in \Xi(\mathcal{X})} \Psi\left(M(\xi, \theta)\right),$$

where $\Xi(\mathcal{X})$ is a set of all possible combinations of design weights and treatment points in $\mathcal{X}$.

Penalty Function

- Penalty function can be used to incorporate measure of component of damages and trial costs into the design process.
- If components are expensive and testing is destructive, one may wish to penalize high stress levels.

Penalized Locally Optimal Design

The design

$$\xi^* = \arg\min_{\xi \in \Xi(\mathcal{X})} \Psi\left(nM(\xi, \theta)\right)$$

subject to the restricted penalty function $n\Phi(\xi) < C$ is equivalent to

$$\xi^* = \arg\min_{\xi \in \Xi(\mathcal{X})} \Psi\left(\frac{M(\xi, \theta)}{\Phi(\xi)}\right).$$

where

- C is a predetermined limit on the total cost of an experiment.
- $\Phi(\xi) = \sum_{i=1}^K w_i\phi(x_i, \theta)$.

Restricted Penalty Function: $\phi(x, \theta, C_1, C_2)$ (Example)

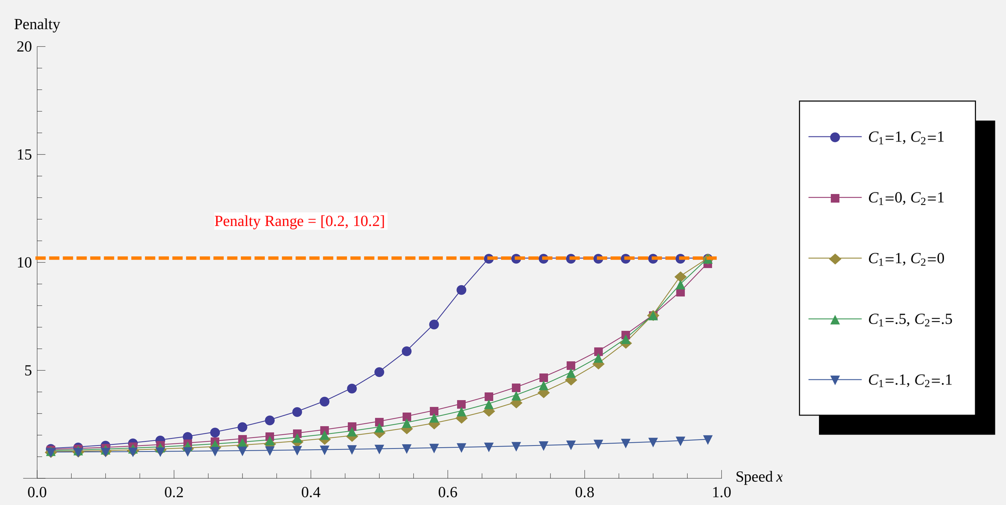


Figure: Restricted penalty function

$$\phi(x, \theta, C_1, C_2) = (1 - p_{1\cdot}(x, \theta))^{-C_1}(1 - p_{\cdot 1}(x, \theta))^{-C_2} + c, \text{ where } c = 0.2,$$
$$\phi(x, \theta, C_1, C_2) \in [0.2, 10.2]$$
$$\theta_0 = -1.5, \theta_1 = 3.6, \theta_2 = 5.4, \theta_3 = 2.7, \sigma = 2, \beta_1 = .1, \beta_2 = .2$$

A Bivariate Weibull Model (David D. Hanagal (2005))

$$f(y, z) = \begin{cases} \beta_1(\beta_5 + \beta_3)\sigma^2(yz)^{\sigma-1}e^{-(\beta_5+\beta_3)z^{\sigma}-(\beta_1+\beta_2-\beta_5)y^{\sigma}} & \text{for } 0 < y < z < \infty \\ \beta_2(\beta_4 + \beta_3)\sigma^2(yz)^{\sigma-1}e^{-(\beta_4+\beta_3)y^{\sigma}-(\beta_1+\beta_2-\beta_4)z^{\sigma}} & \text{for } 0 < z < y < \infty \\ \beta_3\sigma y^{\sigma-1}e^{-(\beta_1+\beta_2+\beta_3)y^{\sigma}} & \text{for } 0 < y = z < \infty. \end{cases}$$

The marginal density functions of the bivariate Weibull density are

$$f(y) = (w_1)\, Weibull((\beta_4 + \beta_3), \sigma) + (1 - w_1)\, Weibull((\beta_1 + \beta_2 + \beta_3), \sigma);$$
$$f(z) = (w_2)\, Weibull((\beta_5 + \beta_3), \sigma) + (1 - w_2)\, Weibull((\beta_1 + \beta_2 + \beta_3), \sigma)$$

where $w_1 = \left(\frac{\beta_2}{\beta_1+\beta_2-\beta_4}\right)$, $w_2 = \left(\frac{\beta_1}{\beta_1+\beta_2-\beta_5}\right)$.

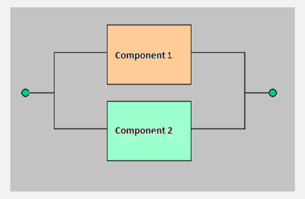
A Bivariate Weibull Regression Model

$$-log(\beta_5 + \beta_3) = \eta_1(x; \theta_1)$$
$$-log(\beta_4 + \beta_3) = \eta_2(x; \theta_2)$$
$$-log(\beta_1 + \beta_2 + \beta_3) = \eta_3(x; \theta_3),$$

where $\eta_1(x; \theta_1)$, $\eta_2(x; \theta_2)$, and $\eta_3(x; \theta_3)$ are functions which in general may be nonlinear; θ_1,θ_2, and θ_3 may be vectors of parameters.

Example: System with Two Parallel Components

Extreme forces or stresses can cause one or two components failure or damage.



(a) Parallel System

(b) Crash Test

A Problem of the Crash Test

- A problem is the crash test is usually performed on just one vehicle under one set of conditions.
- Especially, the ANCAP conducts the frontal offset crash test with just one speed level (64km/h).
- It is unreliable to assess safety on just one vehicle with one speed level for each test because there are several sources of variation.
- To address these problems, a good statistical model for various speeds evaluated with a small number of trials and provided more significant information should be considered to obtain better assessment.

Crash Test

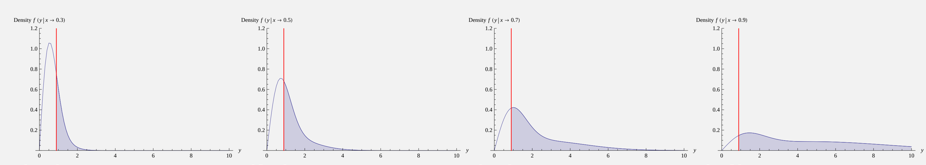
1. Component 1: The human component (human dummy).
2. Component 2: The machine component (vehicle).
3. The frequency of failures (damages) of two components follow the bivariate Weibull density.
4. Only the one covariate: standardized measure of speed

$$x = \frac{\text{Observed speed - Minimum speed}}{\text{Maximum speed - Minimum speed}}.$$

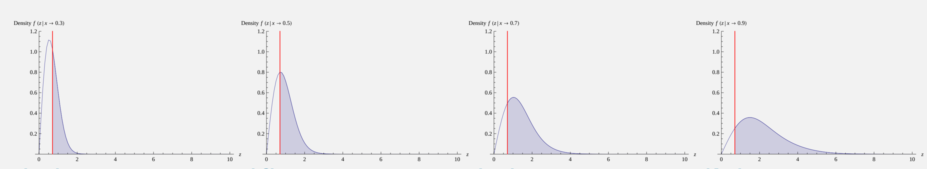
5. The regression equations:
$\eta_i(x; \theta_i) = \theta_0 + \theta_i x$, where i=1, 2, 3.

$$-log(\beta_5 + \beta_3) = \theta_0 + \theta_1 x$$
$$-log(\beta_4 + \beta_3) = \theta_0 + \theta_2 x$$
$$-log(\beta_1 + \beta_2 + \beta_3) = \theta_0 + \theta_3 x.$$

Marginal Weibull Densities of Failure Conditional on Speed x



(a) $f(y|x = .3)$ (b) $f(y|x = .5)$ (c) $f(y|x = .7)$ (d) $f(y|x = .9)$

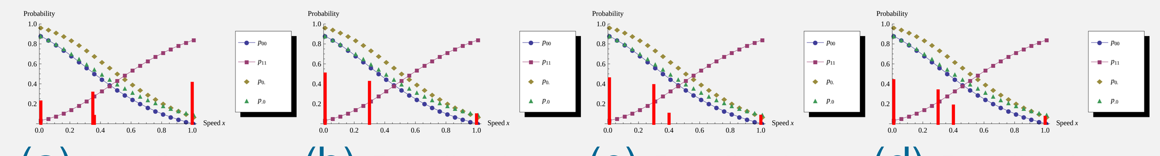


(e) $f(z|x = .3)$ (f) $f(z|x = .5)$ (g) $f(z|x = .7)$ (h) $f(z|x = .9)$

Figure: For the component 1 (Top) for the component 2 (Bottom). (Red vertical bars are cut-off values, $v_1^* = .89443$ (Top),$v_2^* = .70711$ (bottom))

$$\theta_0 = -1.5, \theta_1 = 3.6, \theta_2 = 5.4, \theta_3 = 2.7, \sigma = 2, \beta_1 = .1, \beta_2 = .2$$

Locally D-Optimal & Penalized D-Optimal Designs



(a) Locally D-Optimal (b) Penalized D-Optimal $C_1 = 1, C_2 = 1$ (c) Penalized D-Optimal $C_1 = 0, C_2 = 1$ (d) Penalized D-Optimal $C_1 = 1, C_2 = 0$

Locally D-Optimal Designs with various σs

	Locally D-Optimal Design ξ_D	Efficiency	$\lvert M(\xi_D, \theta)\rvert^{1/m}$
σ=.5	$\left\{\begin{matrix} .20, & .30, & .35, & .36, & .37, & .38, & 1 \\ .1651, & .0426, & .0481, & .1070, & .1180, & .0555, & .0527 \end{matrix}\right\}$	1.1736	.0534
σ=1	$\left\{\begin{matrix} .01, & .35, & .36, & .37, & 1 \\ .2095, & .0176, & .1625, & .2018, & .4085 \end{matrix}\right\}$	1	.0455
σ=2	$\left\{\begin{matrix} .01, & .35, & .36, & 1 \\ .2175, & .3057, & .0728, & .4040 \end{matrix}\right\}$	.7648	.0348
σ=3	$\left\{\begin{matrix} .01, & .34, & .35, & 1 \\ .2269, & .2568, & .1186, & .3977 \end{matrix}\right\}$	.5802	.0264
σ=4	$\left\{\begin{matrix} .01, & .33, & .34, & 1 \\ .2367, & .0895, & .2838, & .3900 \end{matrix}\right\}$	.4330	.0197

$$\text{Efficiency} = \frac{|M(\xi_{D,\sigma}, \theta)|^{1/m}}{|M(\xi_{D,\sigma=1}, \theta)|^{1/m}}.$$

Penalized D-optimal designs with various σs ($C_1 = 1, C_2 = 1$)

	Penalized D-Optimal Design ξ_{PD_1}	Efficiency	$\lvert M(\xi_{PD_1}, \theta)\rvert^{1/m}$	$\left\lvert\frac{M(\xi_{PD_1},\theta)}{\Phi(\xi,\theta)}\right\rvert^{1/m}$
σ=.5	$\left\{\begin{matrix} .01, & .30, & .40, & 1 \\ .4386, & .4621, & .0213, & .0779 \end{matrix}\right\}$	1.1584	.0351	.0159
σ=1	$\left\{\begin{matrix} .01, & .30, & 1 \\ .4574, & .4620, & .0806 \end{matrix}\right\}$	1	.0303	.0131
σ=2	$\left\{\begin{matrix} .01, & .30, & 1 \\ .4977, & .4139, & .0884 \end{matrix}\right\}$	.7954	.0241	.0093
σ=3	$\left\{\begin{matrix} .01, & .30, & 1 \\ .5337, & .3649, & .1014 \end{matrix}\right\}$	.6304	.0191	.0064
σ=4	$\left\{\begin{matrix} .01, & .20, & .30, & 1 \\ .5067, & .2240, & .1507, & .1187 \end{matrix}\right\}$	.4753	.0144	.0042

$$\text{Efficiency} = \frac{|M(\xi_{PD_1,\sigma}, \theta)|^{1/m}}{|M(\xi_{PD_1,\sigma=1}, \theta)|^{1/m}}.$$

Penalized D-optimal designs with various σs ($C_1 = 0, C_2 = 1$)

	Penalized D-Optimal Design ξ_{PD_2}	Efficiency	$\lvert M(\xi_{PD_2}, \theta)\rvert^{1/m}$	$\left\lvert\frac{M(\xi_{PD_2},\theta)}{\Phi(\xi,\theta)}\right\rvert^{1/m}$
σ=.5	$\left\{\begin{matrix} .01, & .30, & .40, & .50, & 1 \\ .3784, & .2991, & .2161, & .0011, & .1053 \end{matrix}\right\}$	1.2232	.0400	.0204
σ=1	$\left\{\begin{matrix} .01, & .30, & .40, & .50, & 1 \\ .3992, & .3902, & .0635, & .0567, & .0904 \end{matrix}\right\}$	1	.0327	.0159
σ=2	$\left\{\begin{matrix} .01, & .30, & .40, & 1 \\ .4494, & .3816, & .0944, & .0747 \end{matrix}\right\}$	.7248	.0237	.0105
σ=3	$\left\{\begin{matrix} .01, & .30, & 1 \\ .4692, & .4461, & .0848 \end{matrix}\right\}$	.5749	.0188	.0073
σ=4	$\left\{\begin{matrix} .01, & .30, & 1 \\ .5012, & .4005, & .0984 \end{matrix}\right\}$	.4495	.0147	.0049

$$\text{Efficiency} = \frac{|M(\xi_{PD_2,\sigma}, \theta)|^{1/m}}{|M(\xi_{PD_2,\sigma=1}, \theta)|^{1/m}}.$$

Penalized D-optimal designs with various σs ($C_1 = 1, C_2 = 0$)

	Penalized D-Optimal Design ξ_{PD_3}	Efficiency	$\lvert M(\xi_{PD_3}, \theta)\rvert^{1/m}$	$\left\lvert\frac{M(\xi_{PD_3},\theta)}{\Phi(\xi,\theta)}\right\rvert^{1/m}$
σ=.5	$\left\{\begin{matrix} .01, & .30, & .60, & 1 \\ .3749, & .4572, & .1110, & .0569 \end{matrix}\right\}$	1.1644	.0347	.0176
σ=1	$\left\{\begin{matrix} .01, & .30, & .50, & 1 \\ .4001, & .4073, & .1305, & .0620 \end{matrix}\right\}$	1	.0298	.0150
σ=2	$\left\{\begin{matrix} .01, & .30, & .40, & 1 \\ .4305, & .3287, & .1767, & .0641 \end{matrix}\right\}$	.7752	.0231	.0115
σ=3	$\left\{\begin{matrix} .01, & .30, & .40, & 1 \\ .4428, & .4180, & .0757, & .0635 \end{matrix}\right\}$	.5940	.0177	.0088
σ=4	$\left\{\begin{matrix} .01, & .30, & 1 \\ .4540, & .4827, & .0632 \end{matrix}\right\}$	.4497	.0134	.0065

$$\text{Efficiency} = |M(\xi_{PD_3,\sigma}, \theta)|^{1/m}/|M(\xi_{PD_3,\sigma=1}, \theta)|^{1/m}.$$