Mixture Design of Quantile Regression Model for Portfolio-Invariant Capital Rule

In Honor of Prof. Angela Dean's Contributions to DOE

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Motivation- Pillard I of *Basel* II Accord

- An international standard that ensures banks have adequate capital for the risks banks exposes itself to, through its lending and investment practices.

--The Basel Committee on
Banking Supervision (BCBS, 2004)

Capital Adequacy of a Bank's Loan Portfolio

Minimum capital requirement of a portfolio
= 99.9% percentile of **the loss *L*** (99.9% VaR)

--The Basel Committee on
Banking Supervision (BCBS, 2004)

Risk Sensitive Capital Allocation

If a loan portfolio has N borrowers with principal weights $w_1,..,w_N$. The **99.9% VaR** must be allocated among the N borrowers so that:

- Borrower i's **marginal capital charge per dollar exposure** θ_i, $1 \leq i \leq N$, should depend on the characteristics of borrower i; instead of the composition of the portfolio, i.e., the exposure weights $w_1,..,w_N$.

Portfolio-Invariant Capital Rule

- The **99.9% VaR** should be a linear function G in the exposure weights, i.e.,

$$99.9\%\ \text{VaR} = G(w_1,\ldots,w_N) = Aw_1\theta_1 + \ldots + Aw_N\theta_N$$

where A is the aggregate principal or exposure,

Model Based Calibration of 99.9% VaR

Portfolio Credit Risk Models in Banking Industry:

- CreditMetrics (RiskMetrics Group),
- CreditRisk+ (Credite Suisse Financial Products),
- Credit Portfolio View (McKinsey),
- Portfolio Manager (KMV).

Merton-type Default Model (Merton, 1974)

- A portfolio has K sectors and each sector has m borrowers. For borrower j in sector k, $1 \leq k \leq K$, $1 \leq j \leq m$, define:

  (1) A_{kj} : the principle size,

  (2) Q_{kj} : the loss per dollar exposure given default (LGD), with mean $E(Q_{kj}) = \mu_k^Q$,

  (3) The borrower defaults with probability $d_{kj} = d_k$

Merton-type Default Model (Merton, 1974)

- The K sectors' risk factors $Z_1, \ldots, Z_K$ are correlated through a **macro risk factor** Z_0 in the form

$$Z_k = \beta_k Z_0 + \sqrt{1-\beta_k^2}\,\eta_k$$

where $0 \le \beta_k \le 1$ is the **correlation parameter** between the sector risk factor Z_k and the macro risk factor Z_0; η_k is a standard normal indep. of Z_0.

Merton-type Default Model (Merton, 1974)

- The financial health index Y_{kj} of borrower j in sector k, $1 \le k \le K$, $1 \le j \le m$, is driven by the sector's systemic risk factor Z_k and an idiosyncratic factor ε_{kj} for the borrower

$$Y_{kj} = \boxed{\lambda_k Z_k + \sqrt{1-\lambda_k^2}\,\varepsilon_{kj}} \sim N(0,1)$$

where the factor loading $0 \le \lambda_k \le 1$.

Merton-type Default Model (Merton, 1974)

- For a borrower in sector k, $1 \leq k \leq K$, default occurs as his financial health index Y_{kj} falls below the threshold

$$\zeta_k = \Phi^{-1}(d_k)$$

d_k is the default probability, Φ is the CDF of a standard normal.

The Aggregate Loss *L*

$$L = A \sum_{1 \le k \le K} \left(\sum_{1 \le j \le m} w_{kj} Q_{kj} 1\{Y_{kj} < \zeta_k\} \right)$$

where

$$A = \sum_{1 \le k' \le K} \sum_{1 \le j' \le m} A_{k'j'}$$

and

$$w_{kj} = \frac{A_{kj}}{A}$$

Infinitely Fine-grained Portfolio

For all sectors,

- The sector’s aggregate exposure sizes $\rightarrow\infty$ as the number of borrowers $m\rightarrow\infty$,
- The ratio of the largest exposure within a sector to the sector’s aggregate exposure vanishes to zero as the number of borrowers $m\rightarrow\infty$.

--Gordy (2003)

Convergence of the Loss *L* (Gordy, 2003)

- Conditional on the risk factors $\boldsymbol{Z}=(Z_1, \ldots, Z_K)$, the aggregate loss L of an infinitely fine-grained portfolio **converges almost surely** to

$$E[L|\boldsymbol{Z}]= A \sum_{1\le k\le K} w_{k\bullet}\mu_k^Q \varphi(Z_k)$$

where $\boldsymbol{w}=(w_{1\bullet},\ldots, w_{K\bullet})$, $w_{k\bullet} = \sum_{1\le j\le m} w_{kj}$

and $\varphi(Z_k) = \Phi\left(\frac{\varsigma_k - \lambda_k Z_k}{\sqrt{1-\lambda_k^2}}\right)$

Infinitely fine-grained portfolio

The **99.9% VaR**, being a function ***G*(*w*)** of the weight exposure ***w*,** is the **99.9% percentile** of the random variable

$$A \sum_{1 \le k \le K} w_k \bullet \mu_k^Q \varphi(Z_k)$$

Sufficient Condition for Portfolio-Invariance: Single Risk Factor (Gordy, 2003)

- When the risk factors $Z_1, \ldots, Z_K$ are all **completely correlated** with the macro risk factor Z_0 i.e., the correlation parameter $\beta_k=1$ for all $1 \leq k \leq K$, then the **99.9% VaR** is

$$G(w) = \boxed{A \sum_{1 \leq k \leq K} w_k \bullet \theta_k}$$

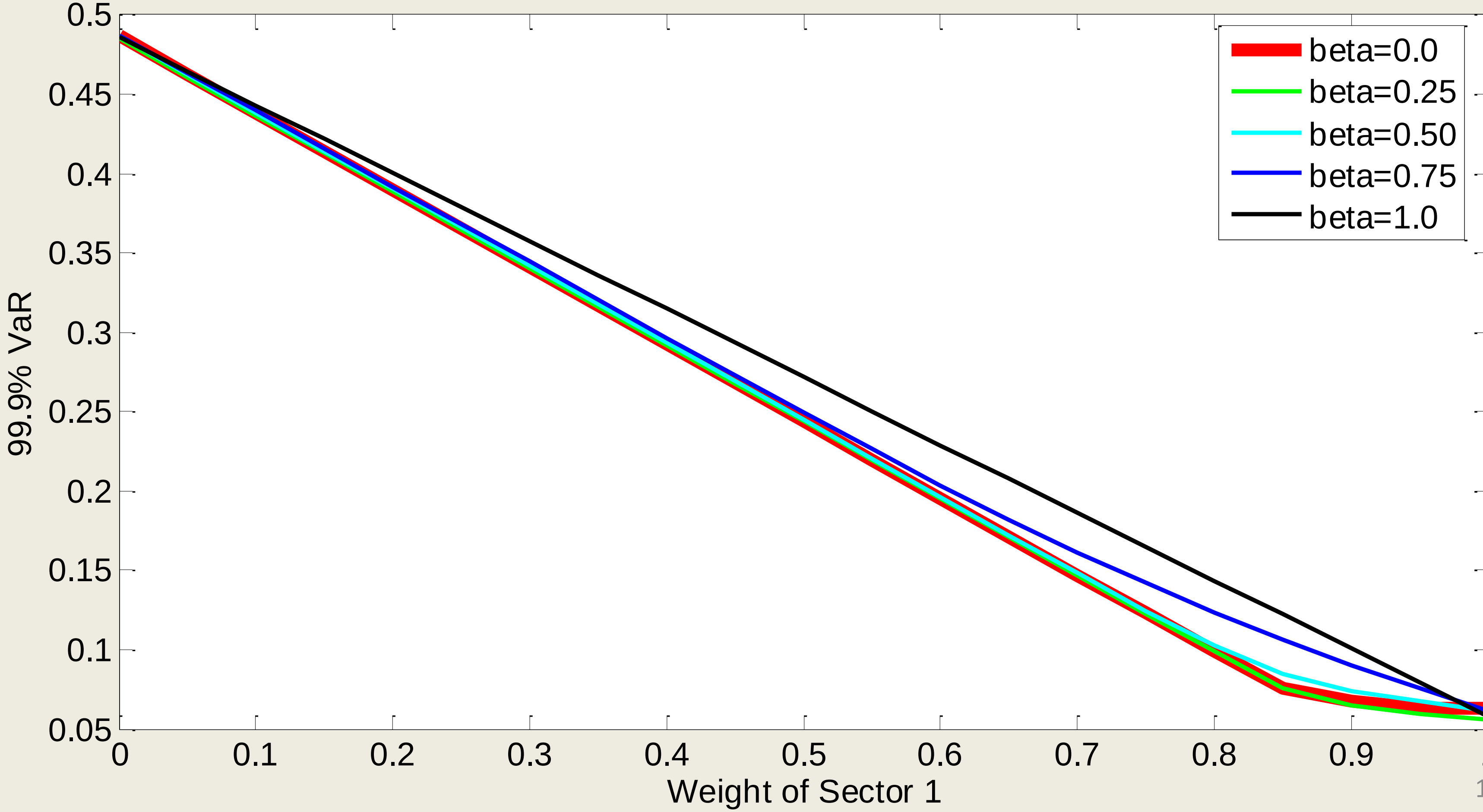
- $q_p^{Z_0}$ is the p^{th} percentile of the macro risk factor Z_0,
- $\theta_k = \mu_k^Q \varphi\left(q_{0.001}^{Z_0}\right)$ is sector k's **marginal capital change per dollar exposure.**

Adjustment of Multiple Risk Factors

- Single risk factor approximation (Pykhtin, 2004),
- Diversification factor (Cespedes and Herrero, 2006),

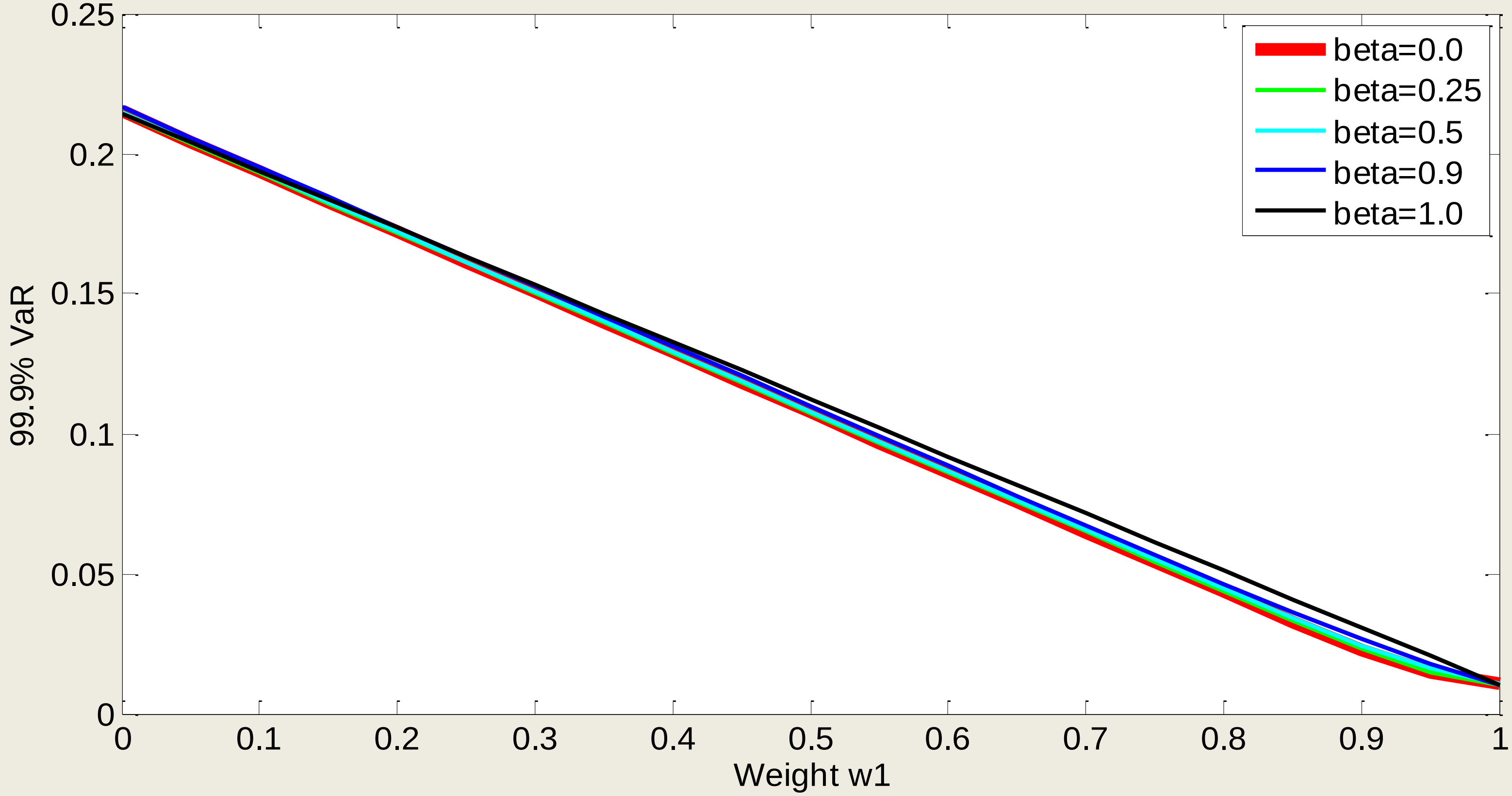
Two-Sector Example

- The default probability d_1=0.01, d_2=0.15,
- The factor loading $\lambda_1=\lambda_2=0.75$,
- The mean of loss given default $E(Q_{kj})$=0.5,



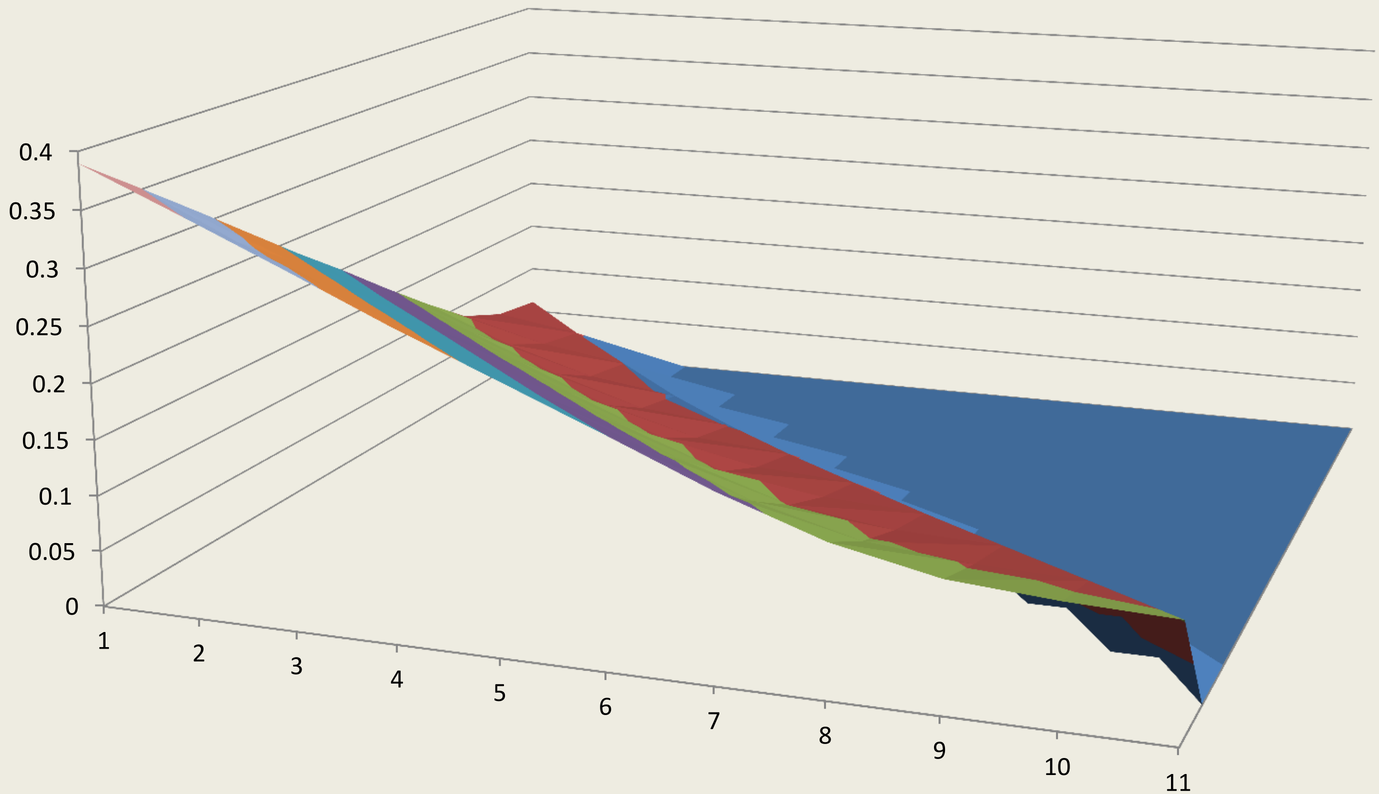
Two-Sector Example

- The default probability d_1=0.01, d_2=0.15,
- The factor loading λ_1=0.3941, λ_2=0.2801 (Lopes, 2004)
- The mean of loss given default $E(Q_{kj})$=0.5,



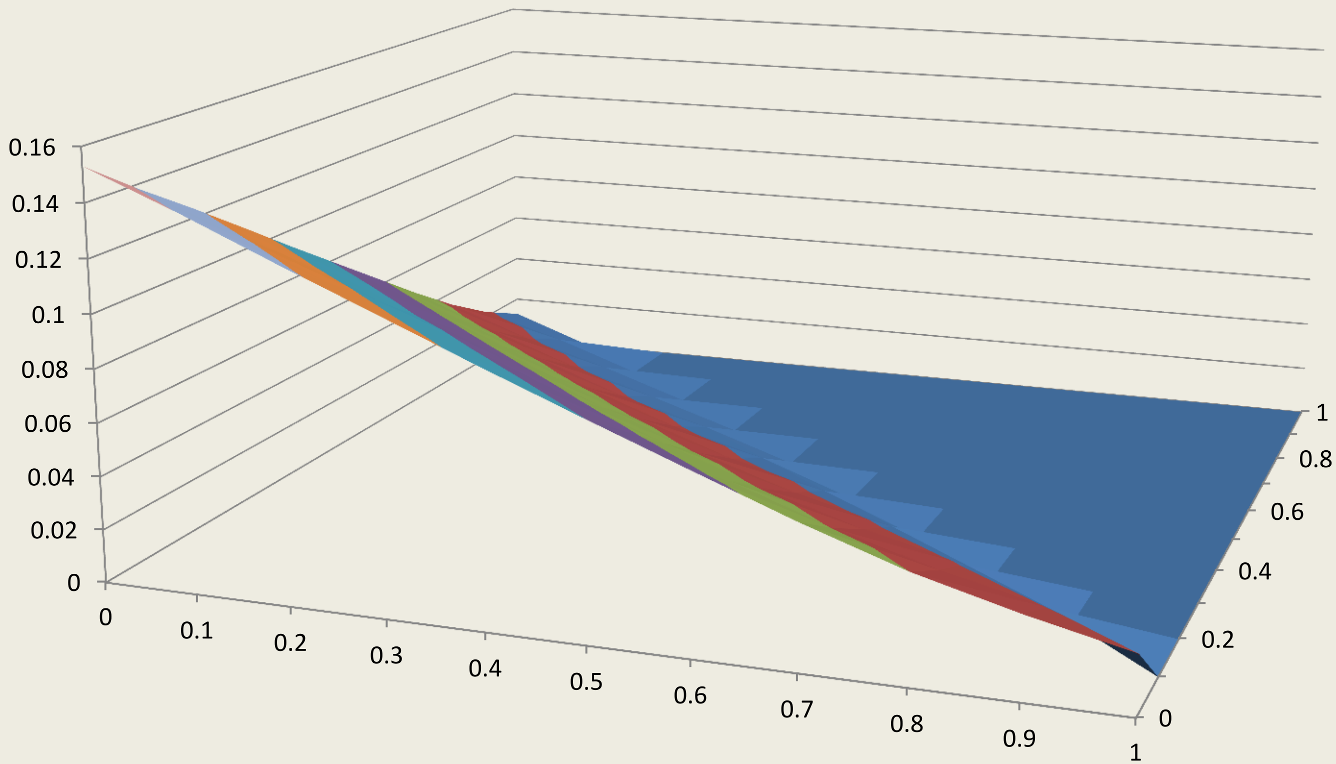
Three-Sector Example

- The default probability d_1=0.0052, d_2=0.0120, d_3=0.1912
- Sector correlation $\beta_1 = \beta_2 = \beta_3$=0.25
- The factor loading $\lambda_1 = \lambda_2 = \lambda_3$=0.5,
- The mean of loss given default $E(Q_{kj})$=0.5,



Three-Sector Example

- The default probability d_1=0.0052, d_2=0.0120, d_3=0.1912
- Sector correlation $\beta_1 = \beta_2 = \beta_3 = 0.25$
- The factor loading λ_1=0.2125, λ_2= 0.1859, λ_3= 0.1200 (Lopes, 2004)
- The mean of loss given default $E(Q_{kj})$=0.5,



Response Surface of Quantile Regression Model

- Choosing *n* portfolio compositions $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_a$, where $\boldsymbol{w}_i = (w^i_{1\bullet}, \ldots, w^i_{K\bullet})'$, $1 \le i \le a$.

- For portfolio composition $\boldsymbol{x}_i$, generating K_i replicates of the risk factors $\boldsymbol{Z}^1, \ldots, \boldsymbol{Z}^{K_i}$, where $\boldsymbol{Z}^j = (Z_{1j}, \ldots, Z_{K,K_j})$, $1 \le j \le K_i$.

- Let $y_{ij} = A \sum_{1 \le k \le K} w_{k\bullet} \mu_k^Q \varphi(Z_k)$, $1 \le j \le K_i$.

Response Surface of Quantile Regression Model

- Assuming the 99.9% percentile $Q(y_{ij})$ is a 2nd degree canonical polynomial

$$Q(y_{ij}) = \sum_{k=1}^{K} \gamma_k w_{k\bullet} + \sum_{k=1}^{K} \sum_{j>k}^{K} \gamma_{jk} w_{k\bullet} w_{j\bullet}$$

- Approximate by a 1st degree canonical polynomial,

$$Q(y_{ij}) = \sum_{k=1}^{K} \theta_k w_{k\bullet}$$

--Koenker and Bassett (1978)

Quantile Regression Model

- Define $\boldsymbol{x}_{i1}' = \boldsymbol{w}_i = (w_{1\bullet}^i, \ldots, w_{K\bullet}^i)'$,

  $\boldsymbol{x}_{i2}' = (w_{1\bullet}^i w_{2\bullet}^i, \ldots, w_{1\bullet}^i w_{K\bullet}^i; \ldots; \ldots, w_{(K-1)\bullet}^i w_{K\bullet}^i)$

- Full Model: $y_{ij} = \boldsymbol{x}_i'\gamma + \varepsilon_{ij}$

  $= \boldsymbol{x}_{i1}'\gamma_1 + \boldsymbol{x}_{i2}'\gamma_2 + \varepsilon_{ij}$, $1 \le i \le n$, $1 \le j \le K_i$

- Reduced Model: $y_{ij} = \boldsymbol{x}_{i1}'\theta_1 + \varepsilon_{ij}$, $1 \le i \le n$, $1 \le j \le K_i$

where the error terms $\varepsilon_{11}, \ldots, \varepsilon_{ij}, \ldots$ are *i.i.d.* with distribution function ***F*** satisfying $\boldsymbol{F^{-1}(p)=0}$ (Koenker and Bassett, 1978).

Least Absolute Deviation Estimation

- The p^{th} quantile regression estimator $\hat{\gamma}$ is the minimizer of the objective function $S(\gamma)$

$$\sum_i \sum_j \left\{ p \sum_{\left\{y_{ij} > \boldsymbol{x}_i{}'\gamma\right\}} \left| y_{ij} - \boldsymbol{x}_i{}'\gamma \right| + (1-p) \sum_{\left\{y_{ij} < \boldsymbol{x}_i{}'\gamma\right\}} \left| y_{ij} - \boldsymbol{x}_i{}'\gamma \right| \right\}$$

--Koenker and Bassett (1978)

Least Absolute Deviation Estimation

- Define the observations $\Gamma_i=\{y_{i1},\ldots, y_{i,K_i}\}$ for the i^{th} design point $\boldsymbol{x}_i$, $1 \le i \le a$,
- y_{i1} is the p^{th} percentile of the subsample Γ_i,
- Define $\boldsymbol{y}_1 = \left(y_{11},\ldots,y_{a1}\right)'$

Least Absolute Deviation Estimation

- **Lemma** (Kao, 2012) Consider a saturated design of full rank $a=K+K(K-1)/2$ in estimating the 2nd degree canonical polynomial. Let the design matrix

$$Z=\left\{\left(\boldsymbol{x}_i'\right)_{i=1}^{a}\right\}$$

The estimator $\hat{\gamma}=Z^{-1}\boldsymbol{y}_1$ minimizes the objective function $S(\gamma)$. In addition, $\hat{\boldsymbol{\gamma}}$ is unique if and only if y_{i1} is the unique p^{th} sample percentile of the subsample $\Gamma_i=\{y_{i1},\dots,y_{i,K_i}\}$ for all $1\leq i\leq a$.

Design Criteria of Fitting a 1st degree Polynomial When the True Model is Quadratic

- The minimization of the mean squared deviation (MSD) between the fitted 1st degree polynomial $\hat{\boldsymbol{y}}$ and the true response surface of 2nd degree canonical polynomial averaged over an experimental region Ω

$$\text{MSD}=\int_{\Omega}\left(\hat{\boldsymbol{y}}-\boldsymbol{x}'\boldsymbol{\gamma}\right)^2\omega(\boldsymbol{x})d\boldsymbol{x}$$

--Box and Draper (1959)

Efficient Design with Minimum MSD

Lemma (Kao, 2012) Suppose a 1st degree polynomial is fitted to the data, when the true model is the 2nd degree canonical polynomial. For large replications K_i, $1 \le i \le n$, there exists a subset $I_1=\{i_1,\ldots,i_q\} \subseteq \{1,\ldots, a\}$, so the optimal design that minimizes MSD minimizes

$$det\left(Z_{21} Z_{11}^{-1} Z_{12} - Z_{22} \right)$$

where

$Z_{11}=\left\{ \left(\boldsymbol{x}_{i1}' \right)_{i \in I_1} \right\}$, $Z_{21}=\left\{ \left(\boldsymbol{x}_{i2}' \right)_{i \in I_1} \right\}$ and $Z_{22}=\left\{ \left(\boldsymbol{x}_{i2}' \right)_{i \notin I_1} \right\}$,

Efficient Mixture Design (Scheffe, 1963)

- For the least squares estimation of:

  (1) 1st degree canonical polynomial, the *A*- and *D*-optimal design is η_1;

  (2) 2nd degree canonical polynomial, the *D*-optimal design is

$$\xi_2^D = \frac{2}{q+1}\eta_1 + \frac{q-1}{q+1}\eta_2$$

  The *A*-optimal design ($q \geq 4$) is

$$\xi_2^A = \frac{\sqrt{4q-3}}{2(q-1)+\sqrt{4q-3}}\eta_1 + \frac{2(q-1)}{2(q-1)+\sqrt{4q-3}}\eta_2$$

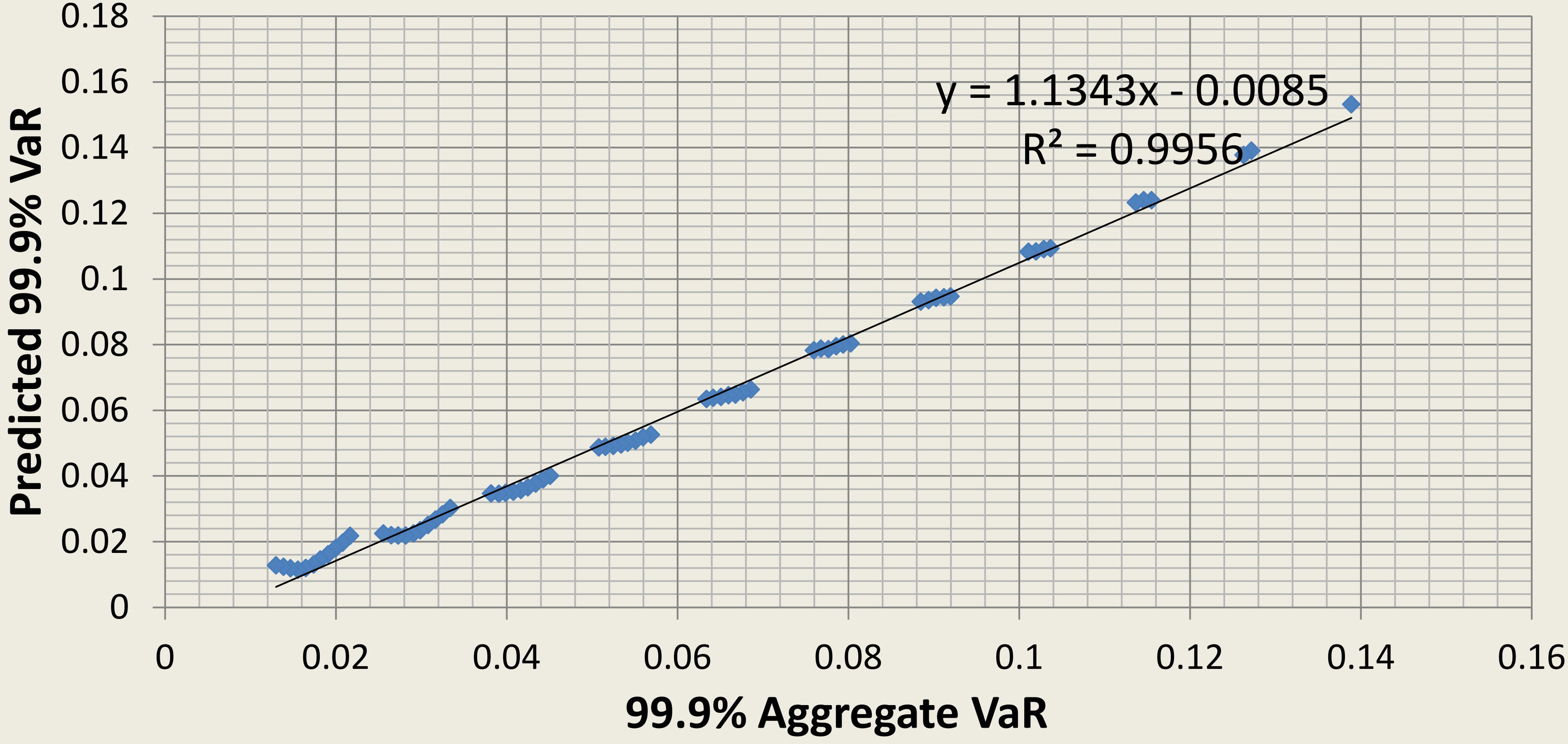
- η_j : The j^{th} elementary centroid design contains all the permutations of the *j*-components (1/*j*, …,1/*j*,…0).

Three-Sector Mixture Design

	Types of Mixture Designs		
	(a) $3\eta_1+3\eta_2$	(b) $2\eta_1+3\eta_2+\eta_3$	(c) $3\eta_1+2\eta_2+\eta_3$
Panel A: The sub-design Z_{11} is $3\eta_1$			
Determinant	1.562×10^{-2}	----	6.944×10^{-3}
MSD	9.349×10^{-3}	----	1.168×10^{-2}
Panel B: The sub-design Z_{11} is $2\eta_1+\eta_2$			
Determinant	3.125×10^{-2}	3.472×10^{-3}	3.472×10^{-3}
MSD	1.332×10^{-2}	3.042×10^{-3}	3.364×10^{-3}
Panel C: The sub-design Z_{11} is $\eta_1+2\eta_2$			
Determinant	6.250×10^{-2}	6.944×10^{-3}	2.778×10^{-2}
MSD	1.069×10^{-2}	1.645×10^{-2}	1.874×10^{-2}
Panel D: The sub-design Z_{11} is $3\eta_2$			
Determinant	6.250×10^{-2}	6.944×10^{-3}	----
MSD	9.496×10^{-3}	7.322×10^{-3}	----
Panel D: The sub-design Z_{11} is $2\eta_2+\eta_3$			
Determinant	----	2.083×10^{-2}	8.333×10^{-2}
MSD	----	1.581×10^{-2}	2.521×10^{-2}

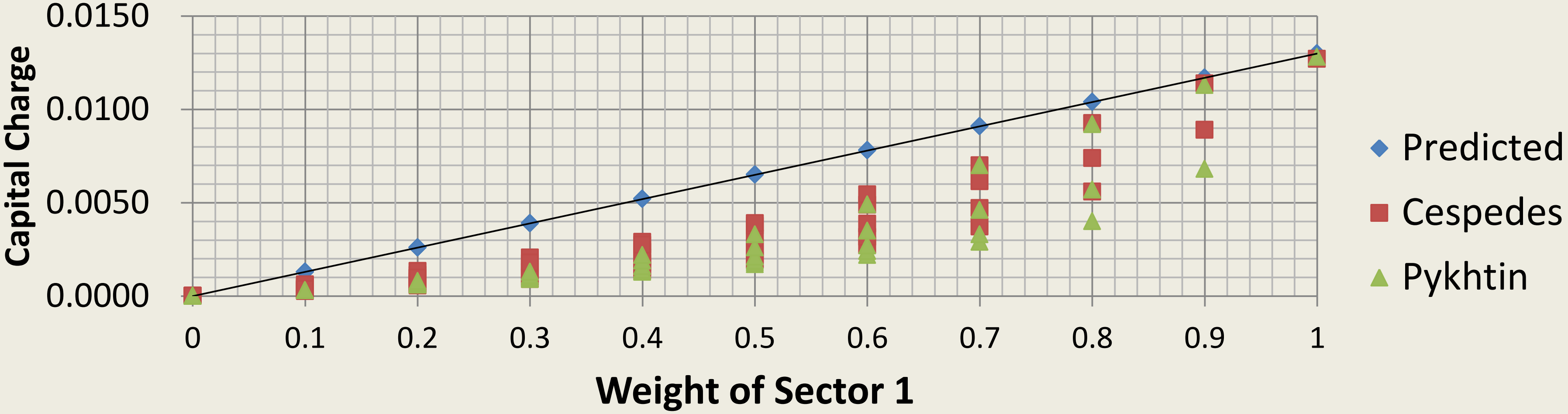
Calibration of the Optimal Design $2\eta_1+3\eta_2+\eta_3$

$$\hat{\theta}_1 = 0.0130, \;\; \hat{\theta}_2 = 0.0217, \; \hat{\theta}_3 = 0.1389$$

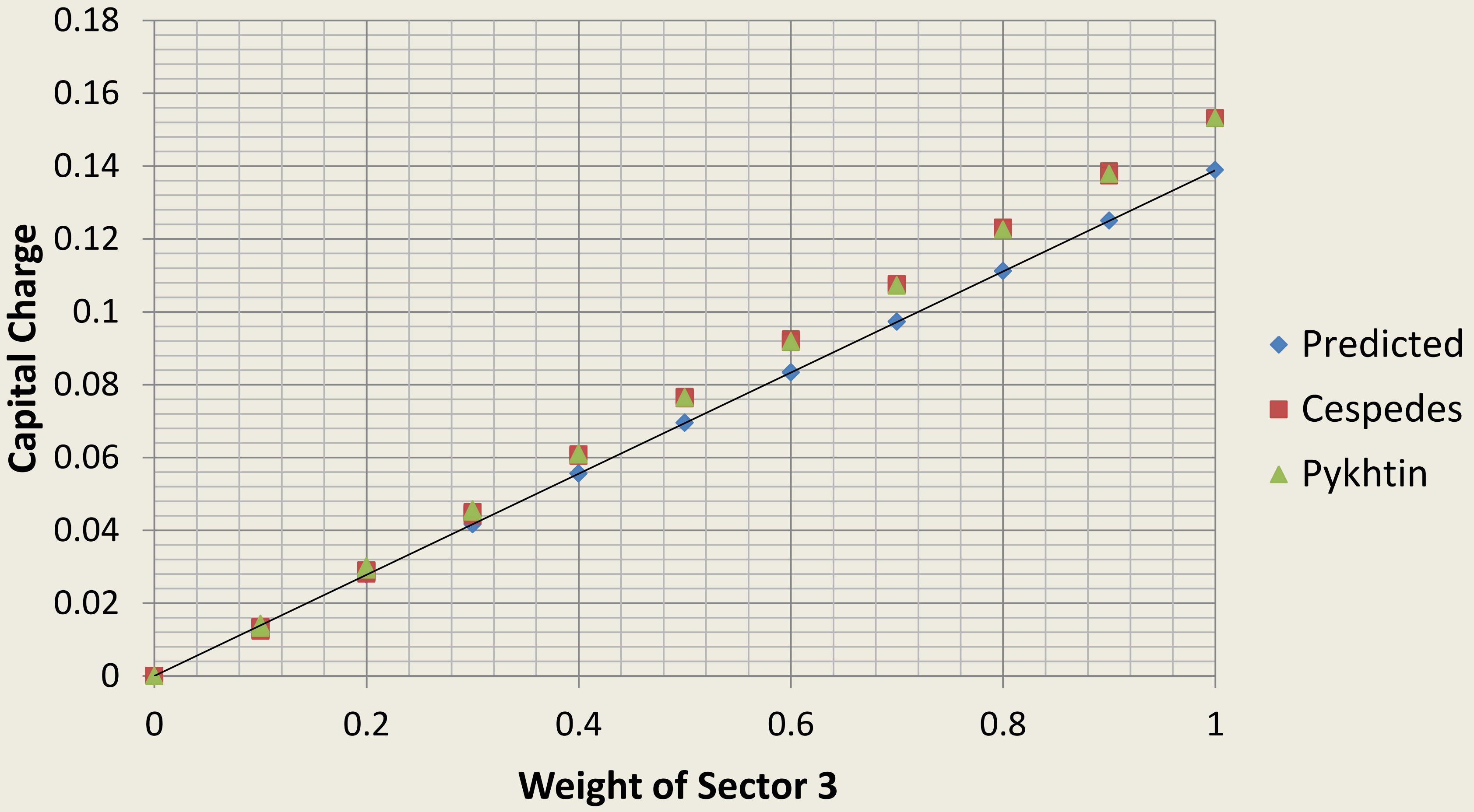


Note. The predicted 99.9% VaR $Q(y_{ij}) = \sum_{k=1}^{3} \hat{\theta}_k w_k$.

Marginal Capital Charge by Sectors One and Two



Marginal Capital Charge by Sector Three



Conclusion

- Till now, estimation of the 99.9% percentile response surface of a 1^{st} degree canonical polynomial based on efficient mixture designs is the only solution for portfolio-invariant capital allocation.

Conclusion

- Optimal design problems have mainly been considered in the context of maximum likelihood or least squares estimation, few results are for quantile regression models,
- Dette, H., M. Trampisch (2013) Optimal designs for quantile regression models, To appear in *Journal of the American Statistical Association*.
- Kao, L.J. (2012) "Efficient Mixture Design Fitting Quadratic Surface with Quantile Responses Using First Degree Polynomial", submitted to *Communications in Statistics-Simulation and Computation*.

Thank You!