Efficient designs for choice experiments with partial profiles

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Outline

1. Introduction
2. Partial profiles
3. Algorithmic designs

1. INTRODUCTION

Choice experiments

Procedure: From each of N **choice sets** of fixed size m choose the 'best' **alternative** or **profile**.

Alternatives are specified by K usually qualitative **attributes**.

Design arranges attribute levels into alternatives and choice sets.

Common model: **Multinomial logistic model (MNL)** for choice probabilities

$$P(\mathbf{x}_{n,i}|C_n) = \frac{\exp[\mathbf{f}(\mathbf{x}_{n,i})^\top \beta]}{\sum_{j=1}^{m} \exp[\mathbf{f}(\mathbf{x}_{n,j})^\top \beta]},$$

where $\mathbf{x}_{n,1}, \ldots, \mathbf{x}_{n,m}$ are the alternatives in the nth choice set C_n and $\mathbf{f}$ is a vector of known regression functions.

Quality of a design ξ for MNL with N choice sets $C_1, \ldots, C_N$ usually assessed by functional of the **information matrix**

$$\mathbf{M}(\xi, \boldsymbol{\beta}) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{X}_n^\top (\mathrm{Diag}(\mathbf{p}_n) - \mathbf{p}_n \mathbf{p}_n^\top) \mathbf{X}_n,$$

where $\mathbf{X}_n$ is a matrix with rows $\mathbf{f}(\mathbf{x}_{n,1})^\top, \ldots, \mathbf{f}(\mathbf{x}_{n,m})^\top$. Further, $\mathbf{p}_n$ is a column vector with elements $P(\mathbf{x}_{n,1}|C_n), \ldots, P(\mathbf{x}_{n,m}|C_n)$.

$\mathbf{M}(\xi, \boldsymbol{\beta})$ depends on the **unknown** parameter vector $\boldsymbol{\beta}$.

Popular criterion: D-optimality, which aims at finding a design ξ that maximizes the determinant of $\mathbf{M}(\xi, \boldsymbol{\beta})$.

Relationship with linear models

Under the **assumption** $\boldsymbol{\beta} = \mathbf{0}$ the matrix $\mathbf{M}(\xi, \boldsymbol{\beta})$ depends **only** on the design ξ.

For example, for N choice sets $C_n = (\mathbf{x}_{n,1}, \mathbf{x}_{n,2})$ of size $m = 2$, that is for **pairs** of alternatives

$$\mathbf{M}(\xi, \boldsymbol{\beta}) = \frac{1}{4N}\mathbf{X}^\top\mathbf{X} = \frac{1}{4}\mathbf{M}(\xi),$$

where $\mathbf{X}$ with rows $[\mathbf{f}(\mathbf{x}_{n,1}) - \mathbf{f}(\mathbf{x}_{n,2})]^\top$ is the design matrix and $\mathbf{M}(\xi)$ the information matrix for the **linear paired comparison model**

$$\boxed{Y(\mathbf{x}_{n,1}, \mathbf{x}_{n,2}) = [\mathbf{f}(\mathbf{x}_{n,1}) - \mathbf{f}(\mathbf{x}_{n,2})]^\top \boldsymbol{\beta} + \varepsilon} \qquad (1)$$

2. PARTIAL PROFILES

Types of profiles

Full profiles ...

... use **all** attributes in every pair

A2	A1
B1	B2
C2	C1

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Full profiles ...

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A2	A1
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Partial profiles ...

... differ only in **subset** of attributes with a given **profile strength**

A3	**A1**
B1	B1
C2	**C1**

Types of profiles

Full profiles ...

... use **all** attributes in every pair

A2	A1
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Partial profiles ...

... differ only in **subset** of attributes with a given **profile strength**

A3	**A1**
C2	**C1**

Setting for PCs of partial profiles

For the remainder of the talk we **assume**

- choice set size $m = 2$,
- $\beta = \mathbf{0}$,
- alternatives are described by K_1 attributes with u_1 levels and K_2 attributes with u_2, levels, where $u_1 < u_2$,
- only **main effects** are to be estimated,
- alternatives in each pair $C_n = (\mathbf{x}_{n,1}, \mathbf{x}_{n,2})$ have different levels for **only** S of the $K = K_1 + K_2$ attributes, where S is the **profile strength**,
- the D-optimality criterion.

GGS (2009) derive optimality results considering the **type** of a set of pairs.

A set of pairs is of type (n_1, n_2) with $n_1 + n_2 = S$, if

for every pair $(\mathbf{x}_1, \mathbf{x}_2)$ in the set the profiles $\mathbf{x}_1$ and $\mathbf{x}_2$ differ in

n_1 attributes with u_1 levels and

n_2 attributes with u_2 levels.

Case	Conditions	Types of pairs in optimal designs
(a)	$K_1, K_2 \geq S$	$(S, 0)$ and $(0, S)$
(b)	$K_2 \geq S > K_1$	$(K_1, S - K_1)$ and $(0, S)$
(c)	$K_1 \geq S > K_2$ and $q_2 S < p$	$(S - K_2, K_2)$ and $(S, 0)$
(d)	$S > K_1, K_2$ and $q_2 S < p$	$(K_1, S - K_1)$ and $(S - K_2, K_2)$
(e)	$S > K_2$ and $q_2 S \geq p$	$(S - K_2, K_2)$

Information matrices of optimal designs

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In the PC model (1), under effects-coding the optimal designs in GGS (2009) have information matrices

$$\mathbf{M}(\xi) = \begin{pmatrix} c_1(\mathbf{I}_{K_1} \otimes \mathbf{M}_{u_1}) & \mathbf{0} \\ \mathbf{0} & c_2(\mathbf{I}_{K_2} \otimes \mathbf{M}_{u_2}) \end{pmatrix}$$

where $\mathbf{M}_a = \frac{2}{a-1}(\mathbf{I}_{a-1} + \mathbf{1}_{a-1}\mathbf{1}_{a-1}^\top)$ and (with $q_i = u_i - 1$, $i = 1, 2$, and $p = K_1 q_1 + K_2 q_2$)

Case	c_1	c_2
(a)-(d)	$q_1 S/p$	$q_2 S/p$
(e)	$1 - (K - S)/K_1$	1

3. ALGORITHMIC DESIGNS

Optimal exact designs

By using **Hadamard** and **weighing matrices**, GGS (2009) also derive optimal designs with **practical** numbers of pairs.

For each of the cases (a)-(e) several **constructions** are proposed such that the corresponding information matrices in the PC model (1) are equal to

$$M(\xi) = \frac{1}{N}\mathbf{X}^\top\mathbf{X} = \frac{1}{N}\begin{pmatrix} \alpha_1(\mathbf{I}_{K_1} \otimes \mathbf{M}_{u_1}) & \mathbf{0} \\ \mathbf{0} & \alpha_2(\mathbf{I}_{K_2} \otimes \mathbf{M}_{u_2}) \end{pmatrix}$$

By imposing appropriate **conditions** on the **building blocks** of the designs the constants α_1 and α_2 become equal to the optimal values c_1 and c_2, which shows that the 'smaller' designs are also **optimal**.

Constructions for exact optimal designs

<table>
<tr><th>ID</th><th>Design matrix $\mathbf{X}$</th><th>Number of pairs N</th></tr>
<tr><td>a1</td><td>$\left(\begin{array}{c|c} \mathbf{1}_{N_1}\otimes\mathbf{W}_1\otimes\mathbf{X}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1}_{N_2}\otimes\mathbf{W}_2\otimes\mathbf{X}_2 \end{array}\right)$</td><td>$K_1N_1C(u_1,2)+K_2N_2C(u_2,2)$</td></tr>
<tr><td>a2</td><td>$\left(\begin{array}{c|c} \mathbf{1}_{N_1}\otimes\mathbf{W}_1\otimes\mathbf{X}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1}_{N_2}\otimes\mathbf{P}_2 \end{array}\right)$</td><td>$K_1N_1C(u_1,2)+N_2\#(\mathbf{P}_2)$</td></tr>
<tr><td>a3</td><td>((a2))</td><td>$K_2N_2C(u_2,2)+N_1\#(\mathbf{P}_1)$</td></tr>
<tr><td>a4</td><td>$\left(\begin{array}{c|c} \mathbf{1}_{N_1}\otimes\mathbf{P}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1}_{N_2}\otimes\mathbf{P}_2 \end{array}\right)$</td><td>$N_1\#(\mathbf{P}_1)+N_2\#(\mathbf{P}_2)$</td></tr>
<tr><td>b1</td><td>$\left(\begin{array}{c|c} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \hline \mathbf{0} & \mathbf{W}_2\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_2 \end{array}\right)$</td><td>$(a_1m_1/z_1+a_2)K_2C(u_2,2)$</td></tr>
<tr><td>b2</td><td>$\left(\begin{array}{c|c} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \hline \mathbf{0} & \mathbf{1}_{a_2}\otimes\mathbf{P}_2 \end{array}\right)$</td><td>$a_1m_1K_2C(u_2,2)/z_1+a_2\#(\mathbf{P}_2)$</td></tr>
<tr><td>b3</td><td>$\left(\begin{array}{c|c} \mathbf{1}_{N_1}\otimes\mathbf{F}_1\otimes\mathbf{1}_{M_1} & \mathbf{1}_{N_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{P}_{2,1} \\ \hline \mathbf{0} & \mathbf{W}_2\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_2 \end{array}\right)$</td><td>$a_1N_1\#(\mathbf{P}_{2,1})+a_2K_2C(u_2,2)$</td></tr>
<tr><td>b4</td><td>$\left(\begin{array}{c|c} \mathbf{1}_{N_1}\otimes\mathbf{F}_1\otimes\mathbf{1}_{M_1} & \mathbf{1}_{N_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{P}_{2,1} \\ \hline \mathbf{0} & \mathbf{1}_{a_2}\otimes\mathbf{P}_2 \end{array}\right)$</td><td>$a_1N_1\#(\mathbf{P}_{2,1})+a_2\#(\mathbf{P}_2)$</td></tr>
<tr><td>c1</td><td>((b1))</td><td>$(a_2m_2/z_2+a_1)K_1C(u_1,2)$</td></tr>
<tr><td>c2</td><td>((b2))</td><td>$a_2m_2K_1C(u_1,2)/z_2+a_1\#(\mathbf{P}_1)$</td></tr>
<tr><td>c3</td><td>((b3))</td><td>$a_2N_2\#(\mathbf{P}_{1,2})+a_1K_1C(u_1,2)$</td></tr>
<tr><td>c4</td><td>((b4))</td><td>$a_2N_2\#(\mathbf{P}_{1,2})+a_1\#(\mathbf{P}_1)$</td></tr>
<tr><td>d1</td><td>$\left(\begin{array}{c|c} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \hline \mathbf{H}_2^{\perp}\otimes\mathbf{W}_{1,2,z_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_1 & \mathbf{H}_2\otimes\mathbf{L}_2\otimes\mathbf{X}_2 \end{array}\right)$</td><td>$m_1n_1C(u_1,2)+m_2n_2C(u_2,2)$</td></tr>
<tr><td>d2</td><td>$\left(\begin{array}{c|c} \mathbf{H}_1\otimes\mathbf{L}_1\otimes\mathbf{X}_1 & \mathbf{H}_1^{\perp}\otimes\mathbf{W}_{2,1,z_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{X}_2 \\ \hline \mathbf{1}_{N_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{P}_{1,2} & \mathbf{1}_{N_2}\otimes\mathbf{F}_2\otimes\mathbf{1}_{M_2} \end{array}\right)$</td><td>$m_1n_1C(u_1,2)+a_2N_2\#(\mathbf{P}_{1,2})$</td></tr>
<tr><td>d3</td><td>((d2))</td><td>$m_2n_2C(u_2,2)+a_1N_1\#(\mathbf{P}_{2,1})$</td></tr>
<tr><td>d4</td><td>$\left(\begin{array}{c|c} \mathbf{1}_{N_1}\otimes\mathbf{F}_1\otimes\mathbf{1}_{M_1} & \mathbf{1}_{N_1}\otimes\mathbf{1}_{a_1}\otimes\mathbf{P}_{2,1} \\ \hline \mathbf{1}_{N_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{P}_{1,2} & \mathbf{1}_{N_2}\otimes\mathbf{F}_2\otimes\mathbf{1}_{M_2} \end{array}\right)$</td><td>$a_1N_1\#(\mathbf{P}_{2,1})+a_2N_2\#(\mathbf{P}_{1,2})$</td></tr>
<tr><td>e1</td><td>$\left(\begin{array}{c|c} \mathbf{H}_2^{\perp}\otimes\mathbf{W}_{1,2,z_2}\otimes\mathbf{1}_{a_2}\otimes\mathbf{X}_1 & \mathbf{H}_2\otimes\mathbf{L}_2\otimes\mathbf{X}_2 \end{array}\right)$</td><td>$m_2n_2C(u_2,2)$</td></tr>
<tr><td>e2</td><td>$\left(\begin{array}{c|c} \mathbf{1}_{a_2}\otimes\mathbf{P}_{1,2} & \mathbf{F}_2\otimes\mathbf{1}_{M_2} \end{array}\right)$</td><td>$a_2\#(\mathbf{P}_{1,2})$</td></tr>
</table>

Conditions on dimensions

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ID	Conditions
a1	$N_1 = N_2 \frac{u_2}{u_1}$
a2	$N_1 = N_2 \frac{u_2}{u_1} \frac{H(S)}{\gcd(K_2,S)}$
a3	$N_1 = N_2 \frac{u_2}{u_1} \frac{\gcd(K_1,S)}{H(S)}$
a4	$N_1 = N_2 \frac{u_2}{u_1} \frac{\gcd(K_1,S)}{\gcd(K_2,S)}$
b1	$n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}, a_2 = a_1 \frac{m_1}{z_1} \left(\frac{p}{q_1 S} - 1\right)$
b2	$n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}, a_2 = a_1 \frac{m_1}{z_1} \left(\frac{p}{q_1 S} - 1\right) \frac{\gcd(K_2,S)}{H(S)}$
b3	$M_1 = \#(\mathbf{P}_{2,1}),\ a_1 = \#(\mathbf{F}_1)$ $a_2 = N_1 u_1 \left(\frac{p}{S} - q_1\right) \frac{H(K_1)H(S-K_1)}{\gcd(K_2,S-K_1)} \left(1 - \frac{u_2 \bmod 2}{2}\right)$
b4	$M_1 = \#(\mathbf{P}_{2,1}), a_1 = \#(\mathbf{F}_1)$ $a_2 = N_1 u_1 \left(\frac{p}{S} - q_1\right) \frac{\gcd(K_2,S)}{\gcd(K_2,S-K_1)} \frac{H(K_1)H(S-K_1)}{H(S)} \left(1 - \frac{u_2 \bmod 2}{2}\right)$
c1	$n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2},\ a_1 = a_2 \frac{m_2}{z_2} \left(\frac{p}{q_2 S} - 1\right)$
c2	$n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2}, a_1 = a_2 \frac{m_2}{z_2} \left(\frac{p}{q_2 S} - 1\right) \frac{\gcd(K_1,S)}{H(S)}$
c3	$M_2 = \#(\mathbf{P}_{1,2}), a_2 = \#(\mathbf{F}_2)$ $a_1 = N_2 u_2 \left(\frac{p}{S} - q_2\right) \frac{H(K_2)H(S-K_2)}{\gcd(K_1,S-K_2)} \left(1 - \frac{u_1 \bmod 2}{2}\right)$
c4	$M_2 = \#(\mathbf{P}_{1,2}), a_2 = \#(\mathbf{F}_2)$ $a_1 = N_2 u_2 \left(\frac{p}{S} - q_2\right) \frac{\gcd(K_1,S)}{\gcd(K_1,S-K_2)} \frac{H(K_2)H(S-K_2)}{H(S)} \left(1 - \frac{u_1 \bmod 2}{2}\right)$
d1	$n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}, n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2}$ $m_2 = m_1 \frac{a_1}{a_2} \frac{z_2}{z_1} \frac{u_2 q_2}{u_1 q_1} \frac{q_1 S - p}{q_2 S - p}$
d2	$M_2 = \#(\mathbf{P}_{1,2}), a_2 = \#(\mathbf{F}_2), n_1 = a_1 \frac{K_2}{z_1} \frac{u_2 q_2}{u_1 q_1}$ $m_1 = N_2 u_1 q_1 \frac{z_1}{a_1} \frac{p - q_2 S}{p - q_1 S} \frac{H(K_2)H(S-K_2)}{\gcd(K_1,S-K_2)} \left(1 - \frac{u_1 \bmod 2}{2}\right)$
d3	$M_1 = \#(\mathbf{P}_{2,1}), a_1 = \#(\mathbf{F}_1), n_2 = a_2 \frac{K_1}{z_2} \frac{u_1 q_1}{u_2 q_2}$

Detailed look at case (a): $K_1, K_2 > S$

Constructions, conditions and numbers of pairs for optimal designs in case (a):

ID	Design matrix $\mathbf{X}$	Conditions	Number of pairs N
a1	$\left(\begin{array}{c\|c} \mathbf{1}_{N_1} \otimes \mathbf{W}_1 \otimes \mathbf{X}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1}_{N_2} \otimes \mathbf{W}_2 \otimes \mathbf{X}_2 \end{array}\right)$	$N_1 = N_2 \frac{u_2}{u_1}$	$K_1 N_1 C(u_1, 2)+$ $K_2 N_2 C(u_2, 2)$
a2	$\left(\begin{array}{c\|c} \mathbf{1}_{N_1} \otimes \mathbf{W}_1 \otimes \mathbf{X}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1}_{N_2} \otimes \mathbf{P}_2 \end{array}\right)$	$N_1 = N_2 \frac{u_2}{u_1} \frac{H(S)}{\gcd(K_2, S)}$	$K_1 N_1 C(u_1, 2) + N_2 \#(\mathbf{P}_2)$
a3	((a2))	$N_1 = N_2 \frac{u_2}{u_1} \frac{\gcd(K_1, S)}{H(S)}$	$K_2 N_2 C(u_2, 2) + N_1 \#(\mathbf{P}_1)$
a4	$\left(\begin{array}{c\|c} \mathbf{1}_{N_1} \otimes \mathbf{P}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1}_{N_2} \otimes \mathbf{P}_2 \end{array}\right)$	$N_1 = N_2 \frac{u_2}{u_1} \frac{\gcd(K_1, S)}{\gcd(K_2, S)}$	$N_1 \#(\mathbf{P}_1) + N_2 \#(\mathbf{P}_2)$

Note: $\mathbf{W}_i$ is weighing matrix order K_i and weight S,
$\mathbf{P}_i$ and $\mathbf{X}_i$ are fixed known matrices,
$\mathbf{1}_{N_i}$ is all-one column vector of length N_i.

Sketch of algorithm for *D*-optimal designs

For illustrative purposes consider only case (a).

1. Check existence of $\mathbf{W}_1$ and $\mathbf{W}_2$.
2. For each of a1–a4 for which the weighing matrices exist:
   2.1 Set $N_2 = 1$.
   2.2 Evaluate condition for N_1.
   2.3 While N_1 is not an integer, increase N_2 by 1. Repeat 2.2.
3. For each of the applicable constructions a1–a4 generate $\mathbf{X}$. Choose the design with the smallest number of pairs.

Remarks:

- The resulting design is *D*-optimal.
- Construction a4 can always be applied.
- Similar but more complicated algorithms exist for the cases (b)–(e).

Parameters of D-optimal designs

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K	K_1	K_2	u_1	u_2	S	ID	N_1	N_2	M_1	M_2	a_1	a_2	m_1	r_1	n_1	s_1	z_1	m_2	r_2	n_2	s_2	z_2	Pairs
4	1	3	2	3	3	b4	1	–	18	–	1	2	–	–	–	–	–	–	–	–	–	–	42
4	2	2	2	3	2	a1	3	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	18
4	2	2	2	3	3	e1	–	–	–	–	–	3	–	–	–	–	–	2	1	2	2	1	12
4	2	2	2	4	2	a1	2	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	16
4	2	2	2	4	3	e1	–	–	–	–	–	3	–	–	–	–	–	4	2	1	1	1	24
4	2	2	2	5	2	a1	5	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	50
4	2	2	2	5	3	e1	–	–	–	–	–	5	–	–	–	–	–	4	2	1	1	1	40
4	2	2	3	4	2	a1	4	3	–	–	–	–	–	–	–	–	–	–	–	–	–	–	60
4	2	2	3	5	2	a1	5	3	–	–	–	–	–	–	–	–	–	–	–	–	–	–	90
4	3	1	2	3	2	c2	–	–	–	–	1	1	–	–	–	–	–	8	1	1	1	1	30
4	3	1	2	3	3	e2	–	–	–	12	–	3	–	–	–	–	–	–	–	–	–	–	36
4	3	1	2	4	2	e1	–	–	–	–	–	2	–	–	–	–	–	2	1	1	1	1	12
4	3	1	2	4	3	e2	–	–	–	12	–	6	–	–	–	–	–	–	–	–	–	–	72
4	3	1	2	5	2	e1	–	–	–	–	–	10	–	–	–	–	–	2	1	3	1	1	60
4	3	1	3	4	2	c2	–	–	–	–	1	2	–	–	–	–	–	2	1	3	1	1	54
5	1	4	2	3	3	b3	1	–	12	–	1	2	–	–	–	–	–	–	–	–	–	–	36
5	2	3	2	3	2	a2	3	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	24
5	2	3	2	3	3	b2	–	–	–	–	1	5	4	2	9	1	1	–	–	–	–	–	96
5	2	3	2	4	2	a2	4	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	44
5	2	3	2	5	2	a2	5	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	70
5	3	2	2	3	2	a3	3	4	–	–	–	–	–	–	–	–	–	–	–	–	–	–	42
5	3	2	2	3	3	c2	–	–	–	–	1	1	–	–	–	–	–	8	2	1	1	1	28
5	3	2	2	4	2	a3	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	18
5	3	2	2	4	3	e1	–	–	–	–	–	2	–	–	–	–	–	4	2	1	1	1	24
5	3	2	3	4	2	a3	2	3	–	–	–	–	–	–	–	–	–	–	–	–	–	–	72
5	3	2	3	4	3	c2	–	–	–	–	2	2	–	–	–	–	–	4	2	3	1	1	96
5	4	1	2	3	2	c1	–	–	–	–	3	3	–	–	–	–	–	2	1	4	1	1	36
5	4	1	2	3	3	e1	–	–	–	–	–	3	–	–	–	–	–	4	1	2	1	2	24
5	4	1	2	4	2	c1	–	–	–	–	1	3	–	–	–	–	–	2	1	2	1	1	28
5	4	1	2	4	3	e1	–	–	–	–	–	3	–	–	–	–	–	4	1	1	1	2	24
5	4	1	2	5	2	e1	–	–	–	–	–	5	–	–	–	–	–	2	1	2	1	1	40
5	4	1	2	5	3	e1	–	–	–	–	–	5	–	–	–	–	–	4	1	1	1	2	40
6	2	4	2	3	2	a1	3	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	30
6	2	4	2	4	2	a1	2	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	28
6	2	4	2	5	2	a1	5	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	90
6	2	4	3	4	2	a1	4	3	–	–	–	–	–	–	–	–	–	–	–	–	–	–	96
6	3	3	2	3	2	a4	3	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	54
6	3	3	2	3	3	a4	3	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	36
6	3	3	2	4	2	a4	2	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	48
6	3	3	2	4	3	a4	2	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	32
6	3	3	2	5	3	a4	5	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	100
6	4	2	2	3	2	a1	3	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	24
6	4	2	2	3	3	c1	–	–	–	–	2	3	–	–	–	–	–	2	1	4	2	1	32
6	4	2	2	4	2	a1	2	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	20
6	4	2	2	4	3	c1	–	–	–	–	2	3	–	–	–	–	–	6	1	2	2	1	80
6	4	2	2	5	2	a1	5	2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	60
6	4	2	2	5	3	e1	–	–	–	–	–	5	–	–	–	–	–	2	1	2	2	1	40
6	4	2	3	4	2	a1	4	3	–	–	–	–	–	–	–	–	–	–	–	–	–	–	84
6	5	1	2	4	2	c2	–	–	–	–	2	6	–	–	–	–	–	2	1	5	1	1	80
6	5	1	2	5	2	c2	–	–	–	–	1	2	–	–	–	–	–	8	1	1	1	1	90

Comparison with JMP10 software

- JMP10 implements so-called unrestricted algorithm of Kessels et al. (2011)
- Allows to include prior information about β, but here $\beta = \mathbf{0}$ ('utility-neutral' designs)
- Comparison for all designs on previous slide with $K = 4$ or $K = 5$
- JMP10 run with
  - 30 random starts in each case (because of run time)
  - options to ignore prior mean and prior variance

Comparison with JMP10 software

Efficiency of JMP10 relative to optimal designs

K	K_1	K_2	u_1	u_2	S	ID	Pairs	D-Efficiency (%)
4	1	3	2	3	3	b4	42	92.60
4	2	2	2	3	2	a1	18	89.44
4	2	2	2	3	3	e1	12	85.17
4	2	2	2	4	2	a1	16	75.45
4	2	2	2	4	3	e1	24	83.32
4	2	2	2	5	2	a1	50	77.15
4	2	2	2	5	3	e1	40	78.93
4	2	2	3	4	2	a1	60	93.76
4	2	2	3	5	2	a1	90	90.26
4	3	1	2	3	2	c2	30	93.36
4	3	1	2	3	3	e2	36	93.51
4	3	1	2	4	2	e1	12	76.56
4	3	1	2	4	3	e2	72	90.42
4	3	1	2	5	2	e1	60	73.38
4	3	1	3	4	2	c2	54	94.27
5	1	4	2	3	3	b3	36	90.94
5	2	3	2	3	2	a2	24	82.69
5	2	3	2	3	3	b2	96	93.00
5	2	3	2	4	2	a2	44	78.97
5	2	3	2	5	2	a2	70	75.53
5	3	2	2	3	2	a3	42	88.89
5	3	2	2	3	3	c2	28	89.49
5	3	2	2	4	2	a3	18	70.21
5	3	2	2	4	3	e1	24	74.28
5	3	2	3	4	2	a3	72	92.62
5	3	2	3	4	3	c2	96	93.71
5	4	1	2	3	2	c1	36	92.27
5	4	1	2	3	3	e1	24	89.81
5	4	1	2	4	2	c1	28	78.28
5	4	1	2	4	3	e1	24	84.21
5	4	1	2	5	2	e1	40	68.89
5	4	1	2	5	3	e1	40	78.97

Consider again case (a).

In order to generate efficient, but not necessarily optimal designs with fewer pairs, only Steps 2. and 3. of the previous algorithm need to be modified:

1. Check existence of $\mathbf{W}_1$ and $\mathbf{W}_2$.
2. For each of a1–a4 for which the weighing matrices exist:
   - 2.1 Set $N_2 = 1$.
   - 2.2 Evaluate condition for N_1. Round N_1 upwards to the nearest integer, generate $\mathbf{X}$ and calculate its efficiency.
   - 2.3 While efficiency is smaller than target value and total number of pairs is smaller than allowed maximum, increase N_2 by 1. Repeat 2.2.
3. Choose the design with the smallest number of pairs and the required efficiency (or make a trade-off).

An example

Design with $N = 10$ pairs for $K_1 = 2$ attributes with $u_1 = 2$ and $K_2 = 2$ attributes with $u_2 = 3$ levels, profile strength $S = 2$.

Optimal design using construction a1 requires $N = 18$ pairs.

a1 with $N_1 = 2$, $N_2 = 1$ and $\mathbf{W}_1 = \mathbf{W}_2 =$ Hadamard order 2:

D-efficiency: 99.06%

Alternative 1				Alternative 2			
1	1	*	*	2	2	*	*
1	2	*	*	2	1	*	*
1	1	*	*	2	2	*	*
1	2	*	*	2	1	*	*
*	*	1	1	*	*	2	2
*	*	1	1	*	*	3	3
*	*	2	2	*	*	3	3
*	*	1	2	*	*	2	1
*	*	1	3	*	*	3	1
*	*	2	3	*	*	3	2

New algorithm by Kessels et al. with ~ 200 starts (> 30min):

D-efficiency: 93.06%

Alternative 1				Alternative 2			
*	*	1	2	*	*	3	1
*	*	2	1	*	*	3	3
*	2	3	*	*	1	2	*
1	2	*	*	2	1	*	*
1	*	2	*	2	*	1	*
*	*	1	1	*	*	3	2
1	*	1	*	2	*	2	*
*	1	*	3	*	2	*	2
*	2	*	3	*	1	*	2
2	*	*	3	1	*	*	1

Concluding remarks

Designs are (locally) optimal for the **multinomial logit model** under assumption that parameters are equal to 0.

Tables of optimal designs with up to 100 pairs are available at:
`http://www.maths.qmul.ac.uk/~hg/PP2G/`

More comprehensive assessment of algorithms needed.

Extension to **three** groups of factors is possible, but more complicated (GGS, accepted).

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