On Sequential Designs Based on Maximum Likelihood Estimation

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Introduction

- Nonlinear models: Information matrix and optimal design ξ^* depend on unknown parameters
- Asymptotic behavior of the maximum likelihood estimator and a sequential design ξ_n are investigated

The Sequential Procedure

Choose Initial Design

- Existence of the estimator depends on initial design!
- Design region $\mathcal{X}$: closed interval in $\mathbb{R}^q$

Observations: A Logistic Binary Response Model

- Sequence of observations: Modeled by binary random variables

$$Y_1, \dots, Y_n, \dots$$

- n-th observation depends on the past through design point x_n only:

$$\mathrm{P}(Y_n = 1) = G(f(x_n)^\top \beta)$$

where $\beta \in \mathbb{R} \times (\mathbb{R}^+)^q$ and $f(x_n) = \left(1\ x_{1n} \cdots x_{qn}\right)^\top$

Estimate: Maximum Likelihood Estimator

- Maximum likelihood estimator based on n observations: $\hat\beta_n$
- If the estimator exists:

$$\hat\beta_n = \hat\beta_{n-1} + \frac{1}{n} M(\xi_n, \hat\beta_{n-1})^{-1} f(x_n)\left(Y_n - G(f(x_n)^\top \hat\beta_{n-1})\right) + H_n$$

- Main component of recursion:
  Yields "mean differential equation" for proof of convergence

Choose a new Design Point

- Information matrix: Fisher Information as in the i.i.d. case

$$M(\xi_n, \beta) = \frac{1}{n} \sum_{i=1}^{n} G(f(x_i)^\top \beta)\left(1 - G(f(x_i)^\top \beta)\right) f(x_i) f(x_i)^\top$$

- Criterion: D-optimality
- New design point: Maximize

$$G(f(x)^\top \hat\beta_n)\left(1 - G(f(x)^\top \hat\beta_n)\right)\ f(x)^\top M(\xi_n, \hat\beta_n)^{-1} f(x)$$

Convergence

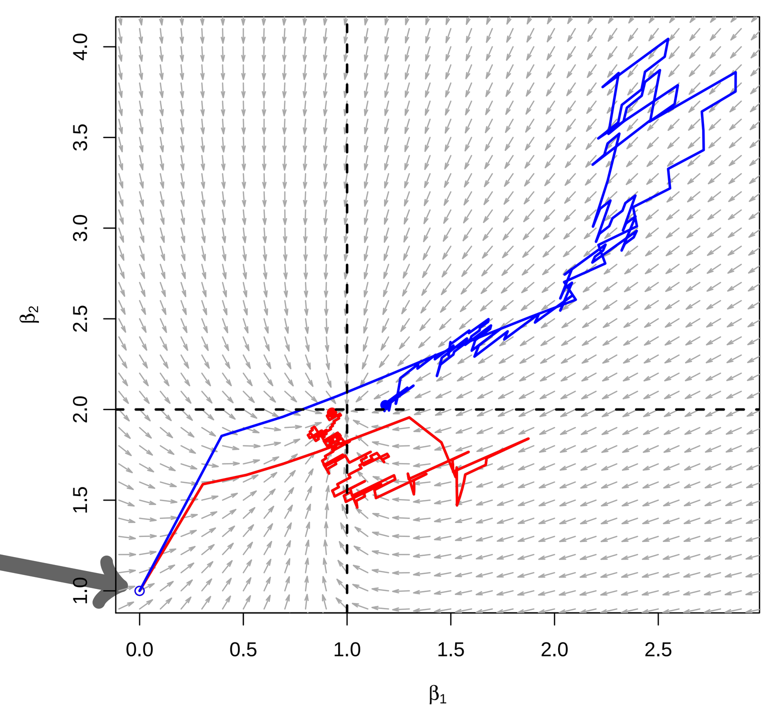
- **Estimator:** $\hat\beta_n \longrightarrow \beta$ **a.s.**
- **Design:** $M(\xi_n, \hat\beta_n) \longrightarrow M(\xi^*, \beta)$

Simulations

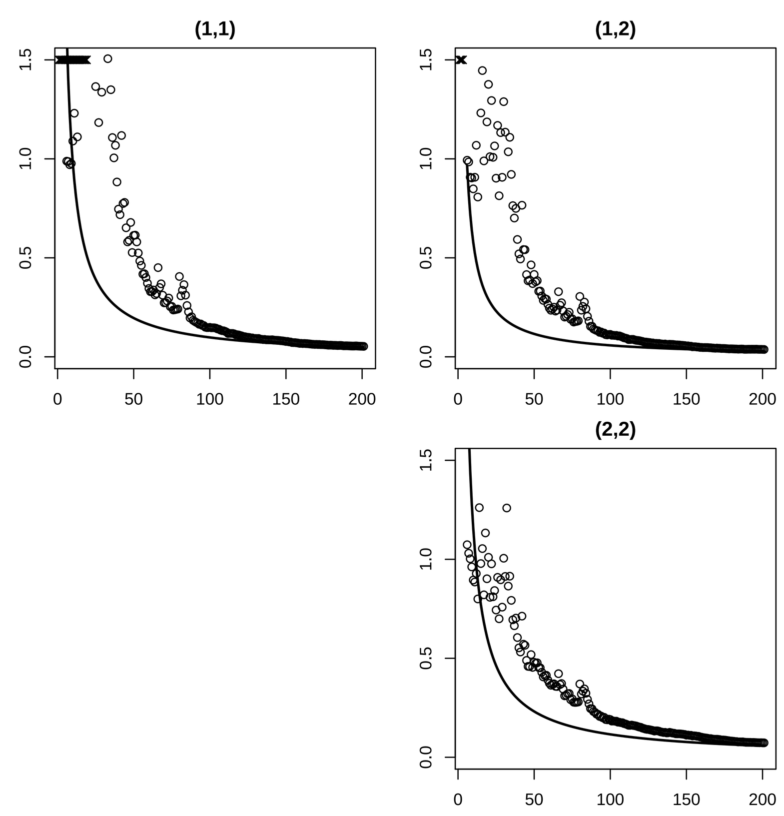
- True parameter: $\beta = \left(1\ 2\right)^\top$
- Design region: $[-10, 10]$
- Number of runs: 500
- Initial design: $0, \pm 1, \pm 5$

Trajectories

- Solid red/blue: Simulated paths
- Dashed lines: True parameter
- Grey arrows: Directional field of the "mean differential equation"
- Initial parameter value



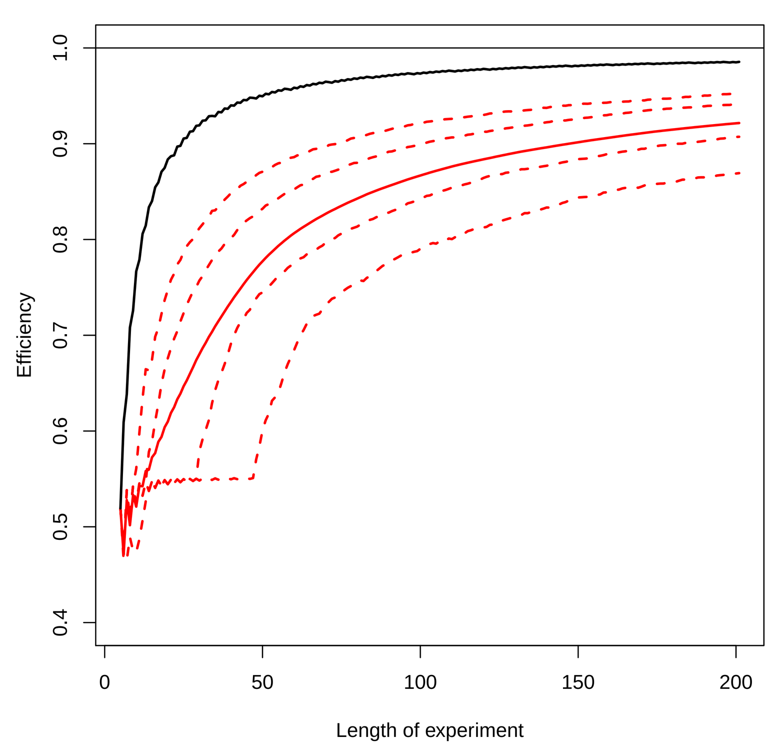
Mean squared error



- Components of simulated MSE-matrix and asymptotic variance
- Circles: Simulated
- Solid line: $\frac{1}{n} M(\xi^*, \beta)^{-1}$

Efficiency

- w.r.t. D-optimal design: balanced, support at $-\mathbf{1.27}$ and $\mathbf{0.27}$



- Solid red: Mean efficiency
- Dashed red: 0.05, 0.1, 0.9, 0.95 quantiles
- Solid black: Wynn-algorithm for known true parameter

Bibliography

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