Fundamental Theorems of Quaternary-code Designs

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Introduction to Quaternary-code (QC) Designs

Regular FFD vs Nonregular FFD vs QCD

	Regular FFDs	Nonregular FFDs	QC Designs
Design Properties	Good enough for most users	Better than regular designs	**Better than regular designs**
Avoid Analytical Pitfalls via Disentanglement	No (Full Aliased Structure)	Yes (Full & Partial Aliased Structure)	**Yes (Full & Partial Aliased Structure)**
Construction Method	Simple	Unknown	**Simple**
Analysis Complication	Simple	Difficult	**Difficult**

Quaternary Codes and Gray Map

(Phoa and Xu, Annals of Statistics 2009)

What is Quaternary Code (QC) Design?

1. QC takes on values from {0,1,2,3} (mod 4), an analogue to Z_2 = {0,1} (mod 2).
2. QC design shares the same linear combination procedure as regular FFDs to construct 4-level designs from the generating matrix over Z_4.
3. A Gray map is used to transform 4-level designs to 2-level designs, which is mostly nonregular.

General Method of Design Construction via Quaternary Code:

$$\text{Generator Matrix } G \text{ (over } Z_4) \xrightarrow{\text{Linear Combination}} \text{4-Level Design } C \text{ (over } Z_4) \xrightarrow{\text{Gray Map}} \text{2-Level Design } D \text{ (over } Z_2)$$

Gray Map:

Z_4	Z_2
0	**(+1,+1)**
1	**(+1,-1)**
2	**(-1,-1)**
3	**(-1,+1)**

General Matrix of QC Designs

Generating Matrix G:

General Structure of Generator Matrix: $G = [V \quad I_n]$

I_n is an $n \times n$ identity matrix with 1 on the diagonal entries.

V is a $n \times k$ matrix over Z_4 .

Example:

$$G = [V \quad I_n]$$

$$G = \begin{bmatrix} \boxed{\begin{matrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{matrix}} & \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} \end{bmatrix}$$

V $\qquad$ I_n

Quaternary-code Even Designs

Example:

$$
G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}
\longrightarrow
C = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 2 & 0 & 0 \\
3 & 3 & 0 & 0 \\
1 & 0 & 1 & 0 \\
2 & 1 & 1 & 0 \\
3 & 2 & 1 & 0 \\
0 & 3 & 1 & 0 \\
2 & 0 & 2 & 0 \\
3 & 1 & 2 & 0 \\
0 & 2 & 2 & 0 \\
1 & 3 & 2 & 0 \\
3 & 0 & 3 & 0 \\
0 & & & \\
1 & & & \\
2 & 3 & 3 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\longrightarrow
D = \begin{bmatrix}
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 & +1 & +1 & +1 & +1 \\
-1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\
-1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 \\
-1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 \\
-1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 \\
+1 & +1 & -1 & +1 & +1 & -1 & +1 & +1 \\
-1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\
-1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 \\
+1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 \\
-1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 \\
+1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 \\
+1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 \\
-1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

64 x 8 Two-level Design

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Trigonometric Approach of QC Designs

(Zhang, Phoa, Mukerjee and Xu, Annals of Statistics 2011)

Quaternary-code Designs via an Trigonometric Approach:

Quaternary-code Method:

$$G = [V \quad I_n] = [v_1, \cdots, v_k, \quad I_1, \cdots, I_n]$$

Gray Map re-definition: For $k = 0,1,2,3$,

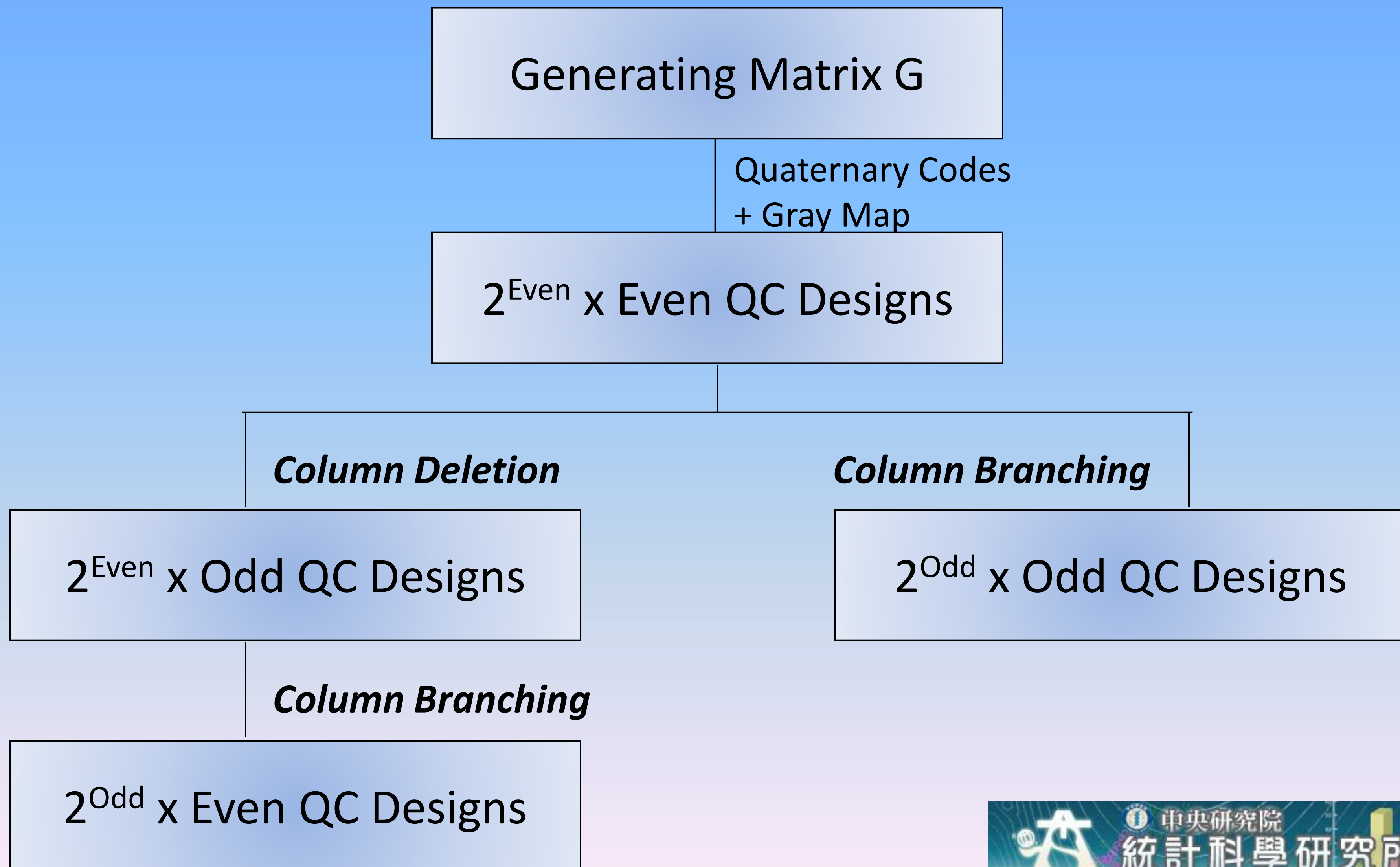
$$k \rightarrow \left(\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{2}k\right), \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\pi}{2}k\right)\right)$$

0 $\rightarrow$ **(+1,+1)**
1 $\rightarrow$ **(+1,-1)**
2 $\rightarrow$ **(-1,-1)**
3 $\rightarrow$ **(-1,+1)**

Z_4 $\qquad$ Z_2

$$D = \sqrt{2}\Big[\sin\left(\frac{\pi}{4} + \frac{\pi}{2}a'v_1\right), \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\pi}{2}a'v_1\right), \cdots, \sin\left(\frac{\pi}{4} + \frac{\pi}{2}a'v_k\right), \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\pi}{2}a'v_k\right)$$
$$\sin\left(\frac{\pi}{4} + \frac{\pi}{2}a_1\right), \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\pi}{2}a_1\right), \cdots, \sin\left(\frac{\pi}{4} + \frac{\pi}{2}a_n\right), \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\pi}{2}a_n\right)\Big]$$

Column Branching and Deletion



$1/4^{th}$-Fraction QC Designs: Comparison

(Phoa and Xu, Annals of Statistics 2009)

Result of the best $1/4^{th}$-Fraction Designs:

$1/4^{th}$-Fraction Design	No. of Factors	No. of Trials	Comparison between QC and Regular Designs		
			Resolution	Aberration	Projectivity
2^{7-2} design	7	32	QC Better	Equivalent	QC Better
2^{8-2} design	8	64	QC Better	Equivalent	QC Better
2^{9-2} design	9	128	Equivalent	Equivalent	QC Better
2^{10-2} design	10	256	QC Better	Equivalent	QC Better
2^{11-2} design	11	512	QC Better	Equivalent	QC Better
2^{12-2} design	12	1024	Equivalent	Equivalent	QC Better
2^{13-2} design	13	2048	QC Better	Equivalent	QC Better
2^{14-2} design	14	4096	QC Better	Equivalent	QC Better
2^{15-2} design	15	8192	Equivalent	Equivalent	QC Better
2^{16-2} design	16	16384	QC Better	Equivalent	QC Better

$1/8^{th}$-Fraction QC Designs: Comparison

(Phoa, Mukerjee and Xu, JSPI 2012)

Result of the best $1/8^{th}$-Fraction Designs:					
$1/8^{th}$-Fraction Design	No. of Factors	No. of Trials	Comparison between QC and Regular Designs		
			Resolution	Aberration	Projectivity
2^{8-3} design	8	32	QC Better	Equivalent	QC Better
2^{9-3} design	9	64	QC Better	Equivalent	QC Better
2^{10-3} design	10	128	QC Better	Equivalent	QC Better
2^{11-3} design	11	256	QC Better	Equivalent	QC Better
2^{12-3} design	12	512	QC Better	Equivalent	QC Better
2^{13-3} design	13	1024	QC Better	Equivalent	QC Better
2^{14-3} design	14	2048	Equivalent	Equivalent	QC Better
2^{15-3} design	15	4096	QC Better	Equivalent	QC Better
2^{16-3} design	16	8192	QC Better	Equivalent	QC Better
2^{17-3} design	17	16384	QC Better	Equivalent	QC Better

A BIG THEORETICAL QUESTION

Given properties of a QC design D generated from $G_{n\times(n+k)} = (V_{n\times k}, I_n)$:

$$\left(\begin{array}{ccc|ccc} v_{11} & \cdots & v_{1k} & I_{11} & \cdots & I_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nk} & I_{n1} & \cdots & I_{nn} \end{array}\right)$$

Determine properties of QC design D' generated from $G_{n\times(n+1+k)} = \left(V_{n\times(k+1)}, I_n\right)$:

$$\left(\begin{array}{cccc|ccc} v_{11} & \cdots & v_{1k} & v_{1(k+1)} & I_{11} & \cdots & I_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nk} & v_{n(k+1)} & I_{n1} & \cdots & I_{nn} \end{array}\right)$$

Fundamental Theorems of Quaternary-code Designs

Code Arithmetic (CA) Approach
(Phoa 2012, in review)

K-equation Labeling in Code Arithmetic Approach:

k-equation in QC designs is similar to wordlengths in Regular FFDs.
Consider $G = (V, I_n)$ and D is generated from G.

$$\boldsymbol{k}_{\vec{\boldsymbol{x}}}$$

$\vec{x} = (x_1, \cdots, x_n)$: n columns in V.
$x_i \in \{0,1,2,3\}$.
0 = None of two columns of D are included.
1 = One of two columns of D is included.
2 = All two columns of D are included.
3 = One of two columns (opposite to 1) is included.

Example: k_{12}:
This is a k-equation for $(1/16)^{th}$-fraction QCD D generated from $G = (v_1, v_2, I_n)$.
Among the first four columns of D, this k-equation considers the subset including columns #1,3,4 or #2,3,4 plus some columns from I_n.

$$\begin{array}{c}
\qquad v_1 \qquad\qquad\qquad v_2 \qquad\qquad\qquad I_n \\
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & \cdots \\
d_{11} & d_{21} & d_{31} & d_{41} & a_{51} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
d_{1n} & d_{2n} & d_{3n} & d_{4n} & a_{5n} & \cdots
\end{pmatrix}
\end{array}$$

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Code Arithmetic (CA) Approach

Arithmetic of k-equations via Code Arithmetic Operator:

$$k_{\vec{w}_1} \oplus k_{\vec{w}_2} = \left(\sum_{\vec{i} \in C(p)} c_{\vec{i}} f_{\vec{i}}\right) \oplus \left(\sum_{\vec{i} \in C(p)} c'_{\vec{i}} f_{\vec{i}}\right) = \sum_{\vec{i} \in C(p)} L(c_{\vec{i}} + c'_{\vec{i}}) f_{\vec{i}}$$

where $L(x)$ is the Lee Weight of x: $L(0) = 0$, $L(1) = 1$, $L(2) = 2$, $L(3) = 1$.

Example: For Quarter-fraction QC designs,

$$k_1 = f_1 + 2f_2 + f_3$$
$$k_2 = 2f_1 + 2f_3$$

Then if we perform $k_1 \oplus k_2$:

$$\begin{aligned} k_1 \oplus k_2 &= (f_1 + 2f_2 + f_3) \oplus (2f_1 + 2f_3) \\ &= L(1+2)f_1 + L(2+2)f_2 + L(1+2)f_3 \\ &= (f_1 + 2f_2 + f_3) = k_3 \end{aligned}$$

$$\begin{pmatrix} & v & & & & I_n & \\ & \downarrow & & \downarrow & & & \downarrow \\ & \downarrow & & \downarrow & & & \downarrow \\ d_1 & & d_2 & a_3 & \cdots & \cdots & a_{2n+2} \end{pmatrix}$$

Code Arithmetic (CA) Approach

Matrix Representation:

Given a QC design D characterized by F, we can always write down

$$\boldsymbol{K = CF}$$

where K are the wordlength equations or k-equations

C is the coefficient matrix for k-equation, or k-matrx.

Example: For Quarter-fraction QC designs,

$$k_1 = f_1 + 2f_2 + f_3$$
$$k_2 = 2f_1 + 2f_3$$
$$\downarrow$$
$$K = CF$$

where $K = (k_1 \quad k_2)^T, \quad F = (f_0 \quad f_1 \quad f_2 \quad f_3)^T$, and

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}.$$

Similarly, for aliasing indexes, we can always write down

$$\boldsymbol{A = BF}$$

where A are the aliasing index equations or a-equations

B is the coefficient matrix for a-equation, or a-matrx.

K-matrix and A-matrix of 1/16th-Fraction QCD

$$K = (k_{01}, k_{10}, k_{02}, k_{11}, k_{13}, k_{20}, k_{12}, k_{21}, k_{22})^T$$

$$F = (f_{00}, f_{01}, f_{02}, f_{03}, f_{10}, f_{11}, f_{12}, f_{13}, f_{20}, f_{21}, f_{22}, f_{23}, f_{30}, f_{31}, f_{32}, f_{33})^T$$

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix}$$

$$A = (a_{01}, a_{10}, a_{11})$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

How to obtain these matrices from the properties of **1/4th-fraction** QC designs via Code Arithmetic approach?

Some Rules on the Structure of QC Designs
(Phoa 2012, in review)

Rule #1:

Given a general k-equation in a $(1/4)^p$th-fraction QC design $k_{\vec{w}} = \sum_{\vec{i} \in C(p)} c_{\vec{i}} f_{\vec{i}}$. Then all k-equations with $w_l = 0$ in a $(1/4)^{p+1}$th-fraction QC design can be expressed as

$$k_{(\vec{w}_l, 0, \vec{w}_{p-l})} = \sum_{s=0}^{3} \sum_{\vec{i} \in C(p)} c_{\vec{i}} f_{(\vec{i}_l, s, \vec{i}_{p-l})}$$

Example:

Given $k_1 = f_1 + 2f_2 + f_3$,

Then:

$$\begin{aligned} k_{10} &= \textstyle\sum_{s=0}^{3} (f_{1s} + 2f_{2s} + f_{3s}) \\ &= f_{10} + f_{11} + f_{12} + f_{13} + 2f_{20} + 2f_{21} + 2f_{22} + 2f_{23} + f_{30} + f_{31} + f_{32} + f_{33}. \end{aligned}$$

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Some Rules on the Structure of QC Designs

(Phoa 2012, in review)

Rule #2:

Given a general k-equation in a $(1/4)^p$ th-fraction QC design $k_{\vec{1}_p} = \sum_{\vec{i}\in C(p)} c_{\vec{i}} f_{(\vec{i}_{p-1}, i_p)}$, where i_p represents the last entry of $\vec{i}$. Then

$$k_{(1_l,\vec{3}_p)} = \sum_{s=0}^{3} \sum_{\vec{i}\in C(p)} c_{\vec{i}} f_{(s,\vec{i}_{p-1},(i_p+s)mod4)}$$

Example:

Given $k_1 = f_1 + 2f_2 + f_3$,

Then:

$$k_{13} = \sum_{s=0}^{3}\left(f_{(s,(1+s)mod4)} + 2f_{(s,(2+s)mod4)} + f_{(s,(3+s)mod4)}\right)$$
$$= f_{01} + f_{12} + f_{23} + f_{30} + 2f_{02} + 2f_{13} + 2f_{20} + 2f_{31} + f_{03} + f_{10} + f_{21} + f_{32}.$$

Some Rules on the Structure of QC Designs

(Phoa 2012, in review)

Rule #3:

Given a general k-equation in a $(1/4)^{p+1}$ th-fraction QC design $k_{\vec{w}} = k_{(\vec{w}_l, s_1, \vec{w}_{p-l})}$. Then for $s_2 = (s_1 + 2) mod 4$,

$$k_{(\vec{w}_l, s_2, \vec{w}_{p-l})} = k_{\vec{w}} \oplus k_{(\vec{0}_l, 2, \vec{0}_{p-l})}$$

Example:

Given $k_{13} = f_{01} + f_{12} + f_{23} + f_{30} + 2f_{02} + 2f_{13} + 2f_{20} + 2f_{31} + f_{03} + f_{10} + f_{21} + f_{32}.$,

Then:

$$\begin{aligned} k_{11} &= k_{13} \oplus k_{02} \\ &= (f_{01} + f_{12} + f_{23} + f_{30} + 2f_{02} + 2f_{13} + 2f_{20} + 2f_{31} + f_{03} + f_{10} + f_{21} + f_{32}) \\ &\quad \oplus (2f_{01} + 2f_{11} + 2f_{21} + 2f_{31} + 2f_{03} + 2f_{13} + 2f_{23} + 2f_{33}) \\ &= \begin{pmatrix} 3f_{01} + f_{12} + 3f_{23} + f_{30} + 2f_{02} + 2f_{11} + 4f_{13} + \\ 2f_{20} + 4f_{31} + 2f_{33} + 3f_{03} + f_{10} + 3f_{21} + f_{32} \end{pmatrix} \\ &= (f_{01} + f_{12} + f_{23} + f_{30} + 2f_{02} + 2f_{11} + 2f_{20} + 2f_{33} + f_{03} + f_{10} + f_{21} + f_{32}) \end{aligned}$$

Some Rules on the Structure of QC Designs
(Phoa 2012, in review)

Rule #4:

Given a general k-equation in a $(1/4)^p$ th-fraction QC design $k_{\vec{w}} = \sum_{\vec{i}_1 \in C(p)} f_{\vec{i}_1} + 2\sum_{\vec{i}_2 \in C(p)} f_{\vec{i}_2}$. Then the a-equation of the corresponding word is $a_{\vec{w}} = \sum_{\vec{i}_1 \in C(p)} f_{\vec{i}_1}$.

Example:

Given $k_{13} = f_{01} + f_{12} + f_{23} + f_{30} + 2f_{02} + 2f_{13} + 2f_{20} + 2f_{31} + f_{03} + f_{10} + f_{21} + f_{32}$,
And $k_{11} = f_{01} + f_{12} + f_{23} + f_{30} + 2f_{02} + 2f_{11} + 2f_{20} + 2f_{33} + f_{03} + f_{10} + f_{21} + f_{32}$,
Their aliasing index equation will be:

$$a_{11} = f_{01} + f_{12} + f_{23} + f_{30} + f_{03} + f_{10} + f_{21} + f_{32}.$$

Given $k_{10} = f_{10} + f_{11} + f_{12} + f_{13} + 2f_{20} + 2f_{21} + 2f_{22} + 2f_{23} + f_{30} + f_{31} + f_{32} + f_{33}$,
Its aliasing index equation will be:

$$k_{10} = f_{10} + f_{11} + f_{12} + f_{13} + f_{30} + f_{31} + f_{32} + f_{33}.$$

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Theorem of QC Designs Structure
(Phoa 2012, in review)

THEOREM:

Given a general k-equation in a $(1/4)^p$th-fraction QC design $k_{\vec{w}}$ where all entries of $\vec{w}$ are odd, the summation of frequencies f_i can be divided into three groups as follows: $k_{\vec{w}} = 0\sum_{\vec{i}_0 \in C(p)} f_{\vec{i}_0} + 1\sum_{\vec{i}_1 \in C(p)} f_{\vec{i}_1} + 2\sum_{\vec{i}_2 \in C(p)} f_{\vec{i}_2}$. Then:

(a) All $\sum \vec{i}_1$ are odd, all $\sum \vec{i}_0$ and $\sum \vec{i}_2$ are even;

(b) There are 2^{2p-1} frequencies with coefficients 1, 2^{2p-2} frequencies with coefficients 0 and 2^{2p-2} frequencies with coefficients 2;

(c) Among those 2^{2p-2} frequencies with coefficients 2, there are 2^{p-1} frequencies that all entries of $\vec{i}_2$ are either 0 or 2;

(d) Same as (c) for those 2^{2p-2} frequencies with coefficients 0.

Example:

Given $k_{13} = f_{01} + f_{12} + f_{23} + f_{30} + 2f_{02} + 2f_{13} + 2f_{20} + 2f_{31} + f_{03} + f_{10} + f_{21} + f_{32}$.,

Then (a) 0+1, 1+2, … ($\sum \vec{i}_1$) are odd, 0+0, 1+1, … ($\sum \vec{i}_0$) and 0+2, 1+3, … ($\sum \vec{i}_2$) are even.

(b) There are 8 frequencies with coefficient 1, 4 frequencies with coefficients 0 & 2.

(c) Among 4 frequencies with coefficient 2, 2 of them are (02, 20).

(d) Among 4 frequencies with coefficient 0, 2 of them are (00, 22).

Necessary and Sufficient Frequencies
(Phoa 2012, in review)

THEOREM:

Given a general $(1/4)^p$th-fraction QC design D. There exists 4^p possible combinations of $\vec{w}$ for k-equations. It is necessary and sufficient to consider the following $\vec{w}$ in order to obtain the properties of D:

(a) all entries are even, except all entries are 0;

(b) the first odd entry must be 1 for $\vec{w}$ that consists of odd entries.

There are $2^p - 1$ frequencies in the first group and $2^{2p-1} - 2^{p-1}$ frequencies in the second group.

Example:

In $(1/4)^{th}$-fraction, among $\{f_0, f_1, f_2, f_3\}$, it is necessary and sufficient to consider:

$$\{f_1, f_2\}$$

In$(1/16)^{th}$-fraction, among $\{f_{00}, f_{01}, f_{02}, f_{03}, f_{10}, f_{11}, f_{12}, f_{13}, f_{20}, f_{21}, f_{22}, f_{23}, f_{30}, f_{31}, f_{32}, f_{33}\}$, it is necessary and sufficient to consider:

$$\{f_{01}, f_{02}, f_{10}, f_{11}, f_{12}, f_{13}, f_{20}, f_{21}, f_{22}\}$$

Extension to 1/64th-Fraction QC Designs
(Phoa 2012, in review; Phoa, Chen and Lin 2012, in review)

$$
C = \begin{pmatrix}
0121012101210121012101210121012101210121012101210121012101210121 \\
0000111122221111000011112222111100001111222211110000111122221111 \\
0000000000000000111111111111111122222222222222221111111111111111 \\
0202020202020202020202020202020202020202020202020202020202020202 \\
0121121021011012012112102101101201211210210110120121121021011012 \\
0121101221011210012110122101121001211012210112100121101221011210 \\
0000222200002222000022220000222200002222000022220000222200002222 \\
0121012101210121121012101210121021012101210121011012101210121012 \\
0121012101210121101210121012101221012101210121011210121012101210 \\
0000111122221111111111112222111100002222111100001111111100001111 \\
0000111122221111111100001111222222221111000011111111222211110000 \\
0000000000000000222222222222222200000000000000002222222222222222 \\
0202111120201111020211112020111102021111202011110202111120201111 \\
0121210101212101012121010121210101212101012121010121210101212101 \\
0202020202020202111111111111111120202020202020201111111111111111 \\
0121121021011012121021011012012121011012012112101012012112102101 \\
0121101221011210101221011210012121011210012110121210012110122101 \\
0121101221011210121001211012210101210112100121101210122101121001 \\
0121121021011012101201211210210121011012012112101210210110120121 \\
0000222200002222111111111111111122220000222200001111111111111111 \\
0121012101210121210121012101210101210121012101212101210121012101 \\
0000111122221111222211110000111100001111222211112222111100001111 \\
0202200200202200200202200200202200200202200200202200200202200202 \\
0202111120201111111111112020111102022020020211110202111111112020 \\
0202111120201111111111110202111120202020020211110202111111112020 \\
0121210101212101121010121210101221010121210101211012121010121210 \\
0121210101212101101212101012121021010121210101211210101212101012 \\
0202020202020202202020202020202002020202020202022020202020202020 \\
0121121021011012210110120121121001211210210110122101101201211210 \\
0121101221011210210112100121101201211012210112102101121001211012 \\
0000222200002222222200002222000000002222000022222222000022220000 \\
0202200200202200111111111111111120200200202202001111111111111111 \\
0202111120201111202011110202111102021111202011112020111102021111 \\
0121210101212101210101212101012101212101012121012101012121010121 \\
0202200200202200202002002020020202022002002022002020020020200202
\end{pmatrix}
$$

$$A = (a_{001}, a_{010}, a_{100}, a_{011}, a_{101}, a_{110}, a_{111})$$

$$
\begin{aligned}
K = \; & (k_{001}, k_{010}, k_{100}, k_{002}, k_{011}, k_{013}, k_{020}, k_{101}, k_{103}, k_{110}, k_{130}, k_{200}, k_{012}, \\
& k_{021}, k_{102}, k_{111}, k_{113}, k_{131}, k_{133}, k_{120}, k_{201}, k_{210}, k_{022}, k_{112}, k_{132}, k_{121}, \\
& k_{123}, k_{202}, k_{211}, k_{213}, k_{220}, k_{122}, k_{212}, k_{221}, k_{222})^T
\end{aligned}
$$

$$
\begin{aligned}
F = \; & (f_{000}, f_{001}, f_{002}, f_{003}, f_{010}, f_{011}, f_{012}, f_{013}, f_{020}, f_{021}, f_{022}, f_{023}, f_{030}, \\
& f_{031}, f_{032}, f_{033}, f_{100}, f_{101}, f_{102}, f_{103}, f_{110}, f_{111}, f_{112}, f_{113}, f_{120}, f_{121}, \\
& f_{122}, f_{123}, f_{130}, f_{131}, f_{132}, f_{133}, f_{200}, f_{201}, f_{202}, f_{203}, f_{210}, f_{211}, f_{212}, \\
& f_{213}, f_{220}, f_{221}, f_{222}, f_{223}, f_{230}, f_{231}, f_{232}, f_{233}, f_{300}, f_{301}, f_{302}, f_{303}, \\
& f_{310}, f_{311}, f_{312}, f_{313}, f_{320}, f_{321}, f_{322}, f_{323}, f_{330}, f_{331}, f_{332}, f_{333})^T
\end{aligned}
$$

$$
B = \begin{pmatrix}
0101010101010101010101010101010101010101010101010101010101010101 \\
0000111100001111000011110000111100001111000011110000111100001111 \\
0000000000000000111111111111111100000000000000001111111111111111 \\
0101101001011010010110100101101001011010010110100101101001011010 \\
0101010101010101101010101010101001010101010101011010101010101010 \\
0000111100001111111100001111000000001111000011111111000011110000 \\
0101101001011010101001011010010101011010010110101010010110100101
\end{pmatrix}
$$

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Properties of 1/64th-Fraction QC Designs
(Phoa 2012, in review; Phoa, Chen and Lin 2012, in review)

THEOREM:

With reference to the $2^{(2n+6)-6}$ QC design D, assume $\sum_{\substack{i,j=odd\\k=even}} f_{ijk} \geq 0$, $\sum_{\substack{i,k=odd\\j=even}} f_{ijk} \geq 0$ and $\sum_{\substack{j,k=odd\\i=even}} f_{ijk} \geq 0$, the following holds:

a) There are $8/{\rho_{100}}^2$ words each with aliasing index ρ_{100}; each ¼ of them have lengths $k_{100}+1$, $k_{120}+3$, $k_{102}+3$ and $k_{122}+5$.
b) There are $8/{\rho_{010}}^2$ words each with aliasing index ρ_{010}; each ¼ of them have lengths $k_{010}+1$, $k_{210}+3$, $k_{012}+3$ and $k_{212}+5$.
c) There are $8/{\rho_{001}}^2$ words each with aliasing index ρ_{001}; each ¼ of them have lengths $k_{001}+1$, $k_{201}+3$, $k_{021}+3$ and $k_{221}+5$.
d) There are $8/{\rho_{110}}^2$ words each with aliasing index ρ_{110}; each ¼ of them have lengths $k_{110}+2$, $k_{130}+2$, $k_{112}+4$ and $k_{132}+4$.

Properties of 1/64th-Fraction QC Designs

(Phoa 2012, in review; Phoa, Chen and Lin 2012, in review)

THEOREM: (continued)

e) There are $8/\rho_{101}{}^2$ words each with aliasing index ρ_{101}; each ¼ of them have lengths $k_{101}+2$, $k_{103}+2$, $k_{121}+4$ and $k_{123}+4$.

f) There are $8/\rho_{011}{}^2$ words each with aliasing index ρ_{011}; each ¼ of them have lengths $k_{011}+2$, $k_{013}+2$, $k_{211}+4$ and $k_{213}+4$.

g) There are $8/\rho_{111}{}^2$ words each with aliasing index ρ_{111}; each ¼ of them have lengths $k_{111}+3$, $k_{113}+3$, $k_{131}+3$ and $k_{133}+3$.

h) There are 7 words each with aliasing index 1; they have lengths $k_{200}+2$, $k_{020}+2$, $k_{002}+2$, $k_{220}+4$, $k_{202}+4$, $k_{022}+4$ and $k_{222}+6$, respectively.

All ρ_{ijk} are defined as $2^{-\lfloor (a_{ijk}+\delta)/2 \rfloor}$, where $\delta=1$ for ρ_{110}, ρ_{101} and ρ_{011}, and $\delta=0$ otherwise.

Optimization of 1/64th-Fraction QC Designs
(Phoa 2012, in review; Phoa, Chen and Lin 2012, in review)

THEOREM:

Given $2^{(2n+6)-6}$ QC design D_0 defined by a frequency vector F_0. Assume D_0 satisfies the conditions in Theorem 3 and it has generalized resolution $R_0 = r_0 + 1 - \rho_0$. Then for $t \geq 0$, a $2^{(2n+126t+6)-6}$ QC design D_t, defined by $F_t = F_0 + \left(0, \vec{1}_{63}\right)t$, has generalized resolution $R_t = r_t + 1 - \rho_t$, where $r_t = r_0 + 64t$ and $\rho_t = \rho_0(2^{-16t})$ if $\rho_0 < 1$ and $\rho_t = 1$ if $\rho_0 = 1$.

Example:

Consider a 256×14 QC design D_0, constructed by $F_0 = \left(\vec{0}_{22}, 1, \vec{0}_2, 1, \vec{0}_5, 1, \vec{0}_7, 1, \vec{0}_{24}\right)$, that has generalized resolution 6.5. Then Theorem 4 suggests that for $t = 1$, a 2^{140-6} QC design D_1 defined by $F_1 = \left(0, \vec{1}_{21}, 2, \vec{1}_2, 2, \vec{1}_5, 2, \vec{1}_7, 2, \vec{1}_{24}\right)$ has $r_1 = 6 + 64(1) = 70$ and $\rho_1 = (1/2)\left(2^{-16(1)}\right) = 2^{-17}$, i.e. generalized resolution 70.9999924.

Summary

Summary

1. The basic formulation of **Quaternary-code Designs (QCD)** is introduced.

2. Some existing results of QCD with **better properties** than regular FFDs are compared.

3. The **Code Arithmetic (CA) approach** provides a systematic notation for k-matrix and a-matrix of QCD.

4. Four rules and two theorems are stated for the **general structure** of QCD.

5. These fundamental theorems are **applied to design properties and optimization** of $(1/64)^{th}$-fraction QCD.

My Selected References related to this Talk

1. **Phoa, F.K.H.** and Xu, H. (2009) Quarter-fraction factorial design constructed via Quaternary codes. *Annals of Statistics*, **37**, 2561-2581.
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Thank You Very Much For Your Attention!

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